The gravitational wave memory effect in linearised gravity and general relativity

Dissertation submitted by

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In partial fulfillment of the

Integrated MSc(5yr) degree in Physics

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May, 2022

Acknowledgements

I am extremely grateful to Prof. Sayan Kar for giving me the opportunity to do my Master's Thesis on Gravitational Wave Memory under his guidance. This work would have been incomplete without his invaluable insights and support. Moreover, my existence in the field of Graviational Wave Astronomy would have not exist without his constant support and guidance since my second year of undergraduate studies. Words will fall short to express my gratitude to him. At last I would like to say that I am lucky enough to have a professor like him during my undergrad studies.

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ABSTRACT

Detection of Gravitational Wave (GW) Memory effect is one of the important future goals among GW researchers. Scientists are looking for various persistent memory observables that are produced by a GW burst. Here I have mostly discussed displacement memory and velocity memory in linearised gravity and in exact radiative spacetimes of General Relativity. The characteristics of memory in the presence of different types of GW burst profiles have been shown in some detail through the work done in this project.

Chapter 1

Introduction

1.1 The gravitational wave memory effect

Gravitational Wave (GW) Memory is a physical phenomenon in four dimensional space-time which can show up in observations related to a GW event. The effect persists even after the wave passes by leaving a permanent imprint on the detector. Because of its persistence and its direct relation with GW bursts, it is named as Gravitational Wave Memory. When the GW burst passes by and influences the positions of say, two test masses, the separation and relative velocity of the test masses change with time leading to persistent displacement memory and persistent relative velocity memory, respectively. This displacement memory can be found by solving for geodesics (and finding the change in separation) or by analysing the geodesic deviation equation. The relative velocity memory is nothing but the derivative of displacement memory w.r.t time. A brief description of several persistent memory observables has been given in Appendix: A [6]. We also study the effect of GW on a ring of particles and notice the permanent change in the shape of the ring caused by a GW burst.

1.2 Status of observations

Several efforts are going on to detect the memory effect but since it is very small in magnitude, scientists have to go a long way to reach the required sensitivity. In Figure 1, we show an expected signal from a GW detector (say LIGO-VIRGO) when memory effect is taken into consideration and caused by the appearance of a GW pulse sandwiched between otherwise flat spacetime regions. If there would have been no memory the signal would have

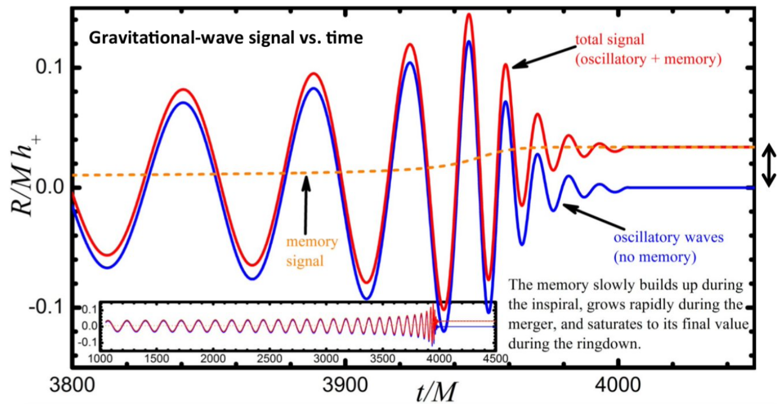

Figure 1.1: Memory effect on the strain data of the GW detector, Courtesy: M. Favata

oscillated around zero but due to memory the positions of the test masses change even after the departure of the pulse. A permanent shift in the GW signal is observed which slowly builds up during the inspiral, grows rapidly during the merger and saturates to its final shifted value during ringdown. Possibility of detection of GW memory with the Advanced LIGO-VIRGO detectors has been discussed in [4].

1.3 Brief overview of work done

In the next section 2.1 I have calculated (following the original work of Braginsky and Grishchuk), geodesic deviation in linearised gravity (weak field solution of Einstein equation). This geodesic deviation equation is solved numerically to obtain displacement memory and velocity memory for different types of GW pulses i.e. Gaussian and double barrier pulse. Thereafter, similar analyses have been done in the exact plane gravitational wave solution of the Einstein equation written using Brinkmann coordinates. Here (i.e. for the exact plane wave), I have solved the geodesic equations directly and obtained memory features.

In Chapter 3, I have shown the analytical solution of the geodesic equation in the exact plane wave spacetime, for a triangular pulse, in both the plus and cross polarised line elements. Some plots of displacement memory have been shown for different values of parameters and the permenant change in shape of a ring of particles has been analysed at different future times for a triangular pulse (in different polarisations).

In Appendix: A [6] we have briefly discussed various persistent memory observables with their integral representations which helps us to write the displacement memory in terms of transverse Jacobi propagators. Appendix B shows the details of the calculations involving a triangular pulse.

Chapter 2

Displacement memory in linearised gravity

2.1 The Braginsky and Grishchuk formalism

The plane wave linearised solution of vacuum Einstein equation in transverse-traceless gauge gives the mathematical description of gravitational wave. A gravitational wave propagating along z direction satisfies the metric

$$ds^2 = -dt^2 + (1 + a(u))dx^2 + (1 - a(u))dy^2 + 2b(u)dxdy + dz^2 \tag{2.1}$$

where $u = t - z$ is retarded time.

The Riemann tensor for the above metric tells us the conditions for the space-time being curved. The only non-zero components of the Riemann tensors that are going to be used for finding geodesic equation are

$$R_{xtxt} = -\frac{1}{2}\ddot{a} \tag{2.2a}$$

$$R_{ytyt} = \frac{1}{2}\ddot{a} \tag{2.2b}$$

$$R_{xtyt} = R_{ytxt} = -\frac{1}{2}\ddot{b} \tag{2.2c}$$

Now using Geodesic deviation equation,

$$\frac{d^2\chi^\alpha}{d\tau^2} = -R^\alpha_{t\beta t}\chi^\beta \tag{2.3}$$

one can find the geodesic deviation of a particle placed at (x,y,z) from origin which is indeed geodesic of the particle in that frame of reference.

Since, $\frac{d^2u}{d\tau^2} = 0$, u is also a affine parameter like proper time τ and we can use u and τ interchangeably whenever needed.

$$u(\tau) = u(\tau_0) + \chi(\tau - \tau_0)$$

where χ is any arbitrary constant.

Now, using (2.3) we get the following differential equations that are satisfied by the geodesics

$$\ddot{x} = -\frac{1}{2}(\ddot{a}x + \ddot{b}y) \tag{2.4a}$$

$$\ddot{y} = -\frac{1}{2}(\ddot{b}x - \ddot{a}y) \tag{2.4b}$$

$$\ddot{z} = 0 \tag{2.4c}$$

We can solve these above differential equations near some values of $t = t_0$ at which the position and velocity of the mass is known. Suppose, a mass is placed at (l^1, l^2, l^3) without any initial velocity at time t=0. Then above equations (2.4) become

$$\begin{aligned}
\ddot{x}(t) &= -\frac{1}{2}(\ddot{a}l^1 + \ddot{b}l^2) \\
\dot{x}(t) &= -\frac{1}{2}[\dot{a}(t) - \dot{a}(0)]l^1 - \frac{1}{2}[\dot{b}(t) - \dot{b}(0)]l^2 + \dot{x}(0) \\
x(t) &= x(0) - \frac{1}{2}[a(t) - a(0)]l^1 - \frac{1}{2}[b(t) - b(0)]l^2 + \frac{1}{2}[\dot{a}(0)l^1 + \dot{b}(0)l^2 + \dot{x}(0)]t
\end{aligned}$$

$$x(t) = l^1 - \frac{1}{2}[a(t) - a(0)]l^1 - \frac{1}{2}[b(t) - b(0)]l^2 + \frac{1}{2}[\dot{a}(0)l^1 + \dot{b}(0)l^2]t \tag{2.5}$$

$$\begin{aligned}
\ddot{y}(t) &= -\frac{1}{2}(\ddot{b}l^1 - \ddot{a}l^2) \\
\dot{y}(t) &= -\frac{1}{2}[\dot{b}(t) - \dot{b}(0)]l^1 + \frac{1}{2}[\dot{a}(t) - \dot{a}(0)]l^2 + \dot{y}(0) \\
y(t) &= y(0) - \frac{1}{2}[b(t) - b(0)]l^1 + \frac{1}{2}[a(t) - a(0)]l^2 + \frac{1}{2}[\dot{b}(0)l^1 - \dot{a}(0)l^2 + \dot{y}(0)]t
\end{aligned}$$

$$y(t) = l^2 - \frac{1}{2}[b(t) - b(0)]l^1 + \frac{1}{2}[a(t) - a(0)]l^2 + \frac{1}{2}[\dot{b}(0)l^1 - \dot{a}(0)l^2]t \quad (2.6)$$

$$\begin{aligned} \ddot{z}(t) &= 0 \\ \dot{z}(t) &= \dot{z}(0) \\ z(t) &= z(0) + \dot{z}(0)t \end{aligned}$$

$$z(t) = l^3 \quad (2.7)$$

2.2 Examples

Now here we will solve the above mentioned geodesic deviation equations (2.4) numerically and will show memory effects[1] between two test masses in presence of a perturbative metric due to a GW burst. We will consider the metric to be plus polarised and using different forms of the function $a(t)$ we will find the displacement and velocity memory.

Gaussian Pulse:

Let's take a Gaussian Pulse of unit amplitude centered at $t = 5$

$$a(t) = e^{-(t-5)^2}$$

and two test masses were placed at (1,1) and (5,5) on x-y plane at t=-50.

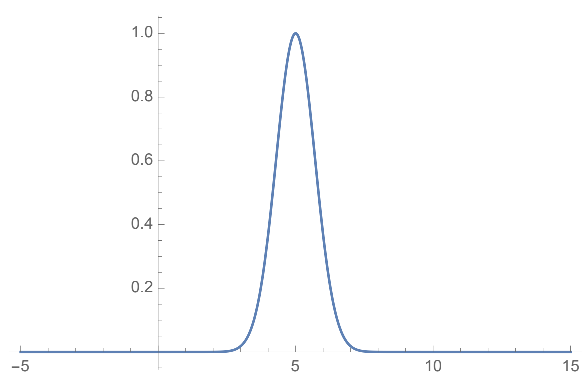

Figure 2.1: Gaussian pulse

The geodesics of the two masses are as follows

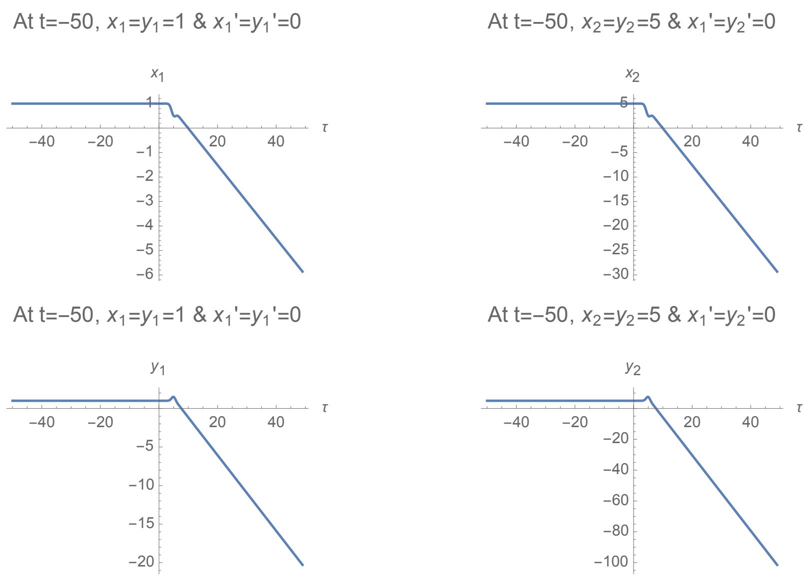

Figure 2.2: Position of the test masses at different times

and corresponding distance between two masses i.e. displacement memory changes accordingly

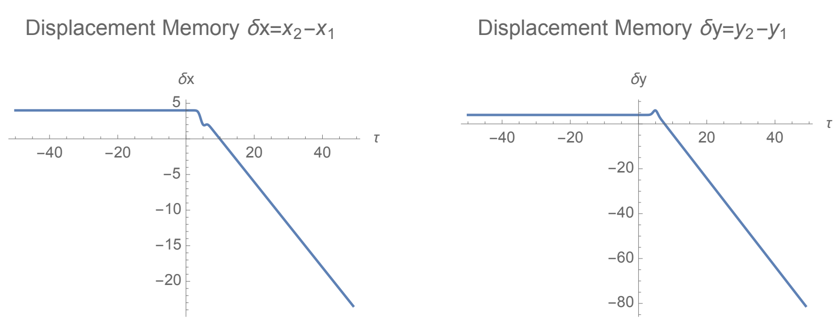

Figure 2.3: Displacement Memory at different times

The velocity of the test masses at different times are shown below and

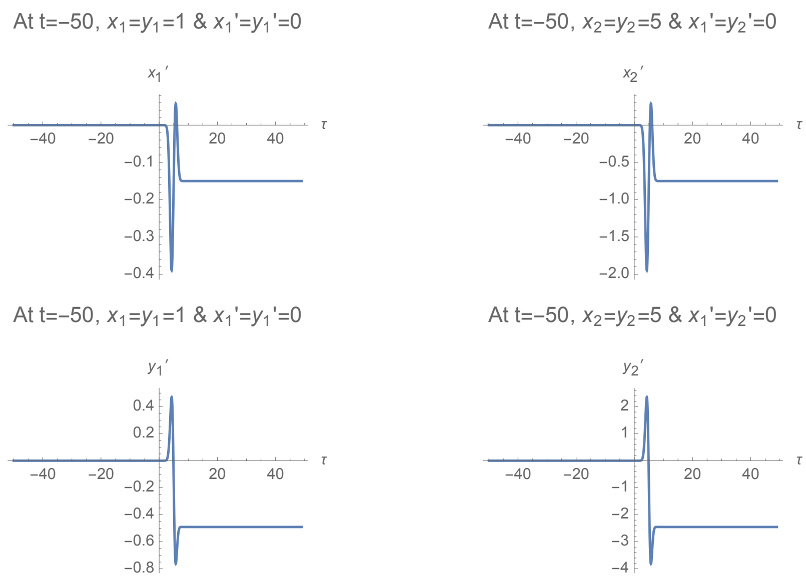

Figure 2.4: Velocity of the test masses at different times

the corresponding velocity memory of the two test masses are as follows

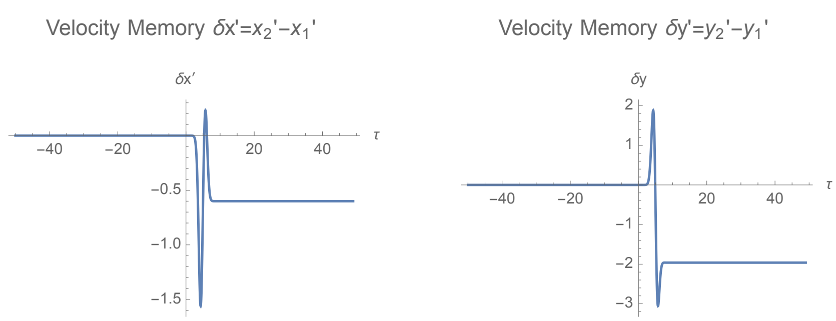

Figure 2.5: Velocity Memory at different times

Double Barrier Pulse

Now we will take double barrier pulse at t=5 as our metric perturbation

$$a(t) = e^{-a(t-5)^2 - \frac{b}{(t-5)^2}}$$

and two test masses were placed at (1,1) and (5,5) on x-y plane at t=-50.

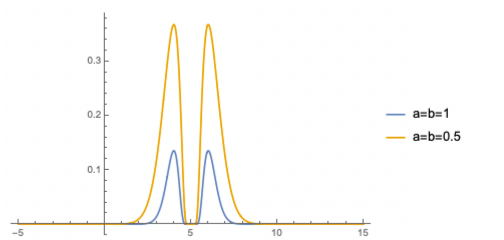

Figure 2.6: Double Barrier pulse

The geodesics of the two masses are as follows

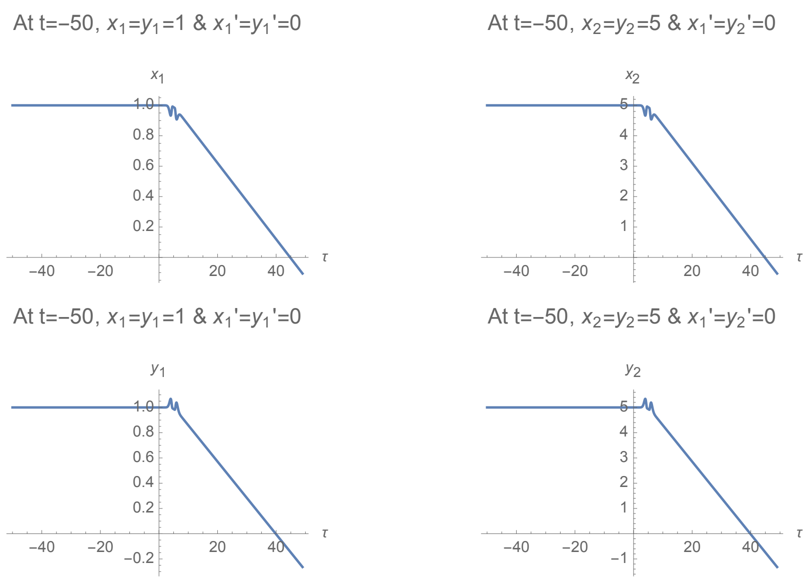

Figure 2.7: Position of the test masses at different times

and corresponding distance between two masses i.e. displacement memory changes accordingly The velocity of the test masses at different times are

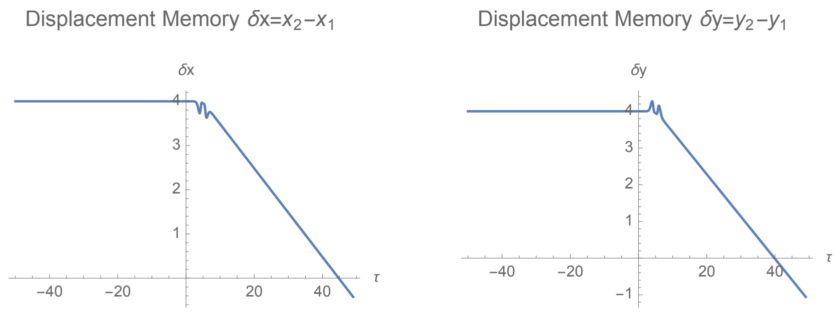

Figure 2.8: Displacement Memory at different times

shown below and the corresponding velocity memory of the two test masses

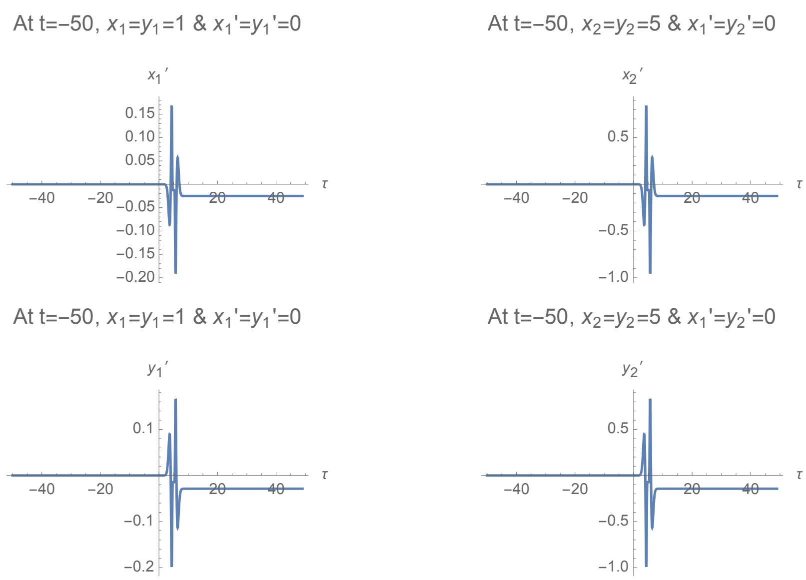

Figure 2.9: Velocity of the test masses at different times

are as follows

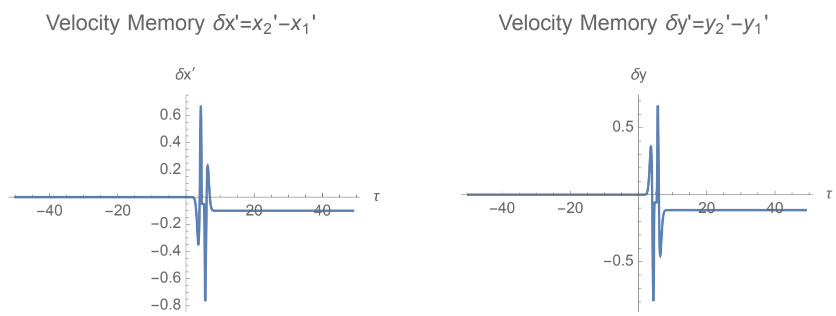

Figure 2.10: Velocity Memory at different times

2.3 Summary of results

For all the above two cases if we notice at the geodesic or displacement memory plots of x and y position of a test mass, the change in x and y coordinate near the GW pulse is opposite in nature which is consistent with the known fact about plus polarisation. From fig 2.3 and 2.8 one can observe the curves cross the horizontal axis which implies two separated test masses collide and after this their relative distance changes it's sign. Since one of it's displacement memory is always inclined towards positive vertical axis it's final velocity after gw passes along x and y direction have opposite signs.

Chapter 3

Memory in exact radiative spacetimes

3.1 The plane fronted exact gravitational wave

In previous section we talked about memory effect in a metric which is linearised approximate solution of source free Einstein equation. Now we will talk about exact nonlinear plane wave solution of Einstein equation[3]. These space-time metrics can be written in terms of Brinkmann coordinates (u, v, x^1, x^2)

$$ds^2 = -2dudv + A_{ij}(u)x^i x^j du^2 + dx^i dx^j \delta_{ij} \tag{3.1}$$

where u and v are retarded time (or phase of the gravitational wave) and advanced time respectively and A_{ij} denotes the GW pulse profile. To satisfy vacuum Einstein equation A_{ij} needs to be symmetric traceless. Here tranverse components of the GW are expressed by x^i where i ranges from 1 to 2.

Now in order to find a killing vector of our interest, we will be looking for a covariantly constant null vector l^a

$$\nabla_b l^a = 0 \tag{3.2}$$

This vector l^a in Brinkmann coordinates can be expressed as

$$l^a = -(\partial_v)^a = (0, -1, 0, 0)$$

Since l^a satisfies

$$\nabla_a l_b + \nabla_b l_a = 0$$

l^a is a killing vector.

3.2 Displacement memory from geodesics

Suppose a test mass follows a geodesic γ which is affinely parametrized by proper time τ. So, using the property of a killing vector we can define a arbitrary parameter χ which follows

$$\dot{\gamma} \cdot l = \dot{\gamma}^a l_a = \dot{u} = \chi \tag{3.3}$$

where dot ($\dot{}$) above some vector represents differentiation w.r.t τ. Now we will define a Lagrangian function from (3.1) to get the geodesics

$$\mathcal{L} = -2\dot{u}\dot{v} + A_{ij}(u)x^i x^j \dot{u}^2 + \dot{x}^i \dot{x}^j \delta_{ij} \tag{3.4}$$

Before proceeding further let's denote the coordinates of γ are $(u(\tau), v(\tau), x^i(\tau))$ at some time τ. We can also obtain the above geodesic equation (3.3) in the following manner using the Lagrangian

$$\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{v}}\right) &= \frac{\partial \mathcal{L}}{\partial v} \\
-2\frac{d}{d\tau}(\dot{u}) &= 0 \\
\dot{u} &= \chi
\end{aligned}$$

$$u(\tau') = u(\tau) + \chi(\tau' - \tau) \tag{3.5}$$

Similarly,

$$\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) &= \frac{\partial \mathcal{L}}{\partial u} \\
-2\ddot{v} + \frac{d}{d\tau}\left(2A_{ij}x^i x^j \dot{u}\right) &= \frac{\partial A_{ij}}{\partial u} x^i x^j \dot{u}^2
\end{aligned}$$

Now if we consider $\chi = 0$, the above equation gives

$$\ddot{v} = 0 \tag{3.6}$$

So, we can say both u and v are affine parameter when $\chi = 0$. Otherwise, only u is the affine parameter for any values of χ. For x^i's we get

$$\begin{aligned}
\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^k}\right) &= \frac{\partial \mathcal{L}}{\partial x^k} \\
2\delta_{ik}\ddot{x}^i &= (A_{ik} + A_{ki})x^i \dot{u}^2 \\
\delta^{kj}\delta_{ik}\ddot{x}^i &= \delta^{kj} A_{ki} x^i \dot{u}^2 \\
\ddot{x}^j &= A^j_i x^i \dot{u}^2
\end{aligned}$$

$$\ddot{x}^i(\tau) = \chi^2 A^i_j(u) x^j(\tau) \tag{3.7}$$

3.3 Displacement memory from geodesic deviation

In Brinkmann metric only non-zero value (i.e. other non-zero terms can be obtained using symmetry/anti-symmetry relation) of Riemann curvature tensor is

$$R^i_{0j0} = -A^i_j(u) \tag{3.8}$$

Now, using the geodesic deviation relation we get

$$\begin{aligned} \frac{d^2\eta^i}{d\tau^2} &= -R^i_{\alpha j\beta}\dot{\gamma}^\alpha \eta^j \dot{\gamma}^\beta \\ &= -R^i_{0j0}\dot{\gamma}^0 \eta^j \dot{\gamma}^0 \\ &= A^i_j(u)\eta^j \dot{u}^2 \end{aligned}$$

$$\frac{d^2\eta^i}{d\tau^2} = \chi^2 A^i_j(u)\eta^j \tag{3.9}$$

where η is the deviation vector. Now, if we consider the deviation of geodesic $\gamma(\tau)$ from origin it reduces to (3.7).

3.4 Examples

Now in this section we will solve geodesic equation (3.7) numerically to show the displacement and velocity memory between two test masses initially placed at $(1,1)$ and $(5,5)$ at time $t = -50$ in $x^1 - x^2$ plane for $\chi = 1$ and different forms of A_{ij}. For simplicity we will be considering plus polarised terms of A_{ij} only. Here for ease we will represent x^1 and x^2 by x and y respectively.

Gaussian Pulse

Here we have chosen the GW pulse profile to be Gaussian in nature around $u = 5$

$$A_{ij}(u) = \left(\begin{array}{cc} e^{-(u-5)^2} & 0 \\ 0 & -e^{-(u-5)^2} \end{array} \right)$$

For this A_{ij} the geodesics of the two test masses are shown above 3.1 and

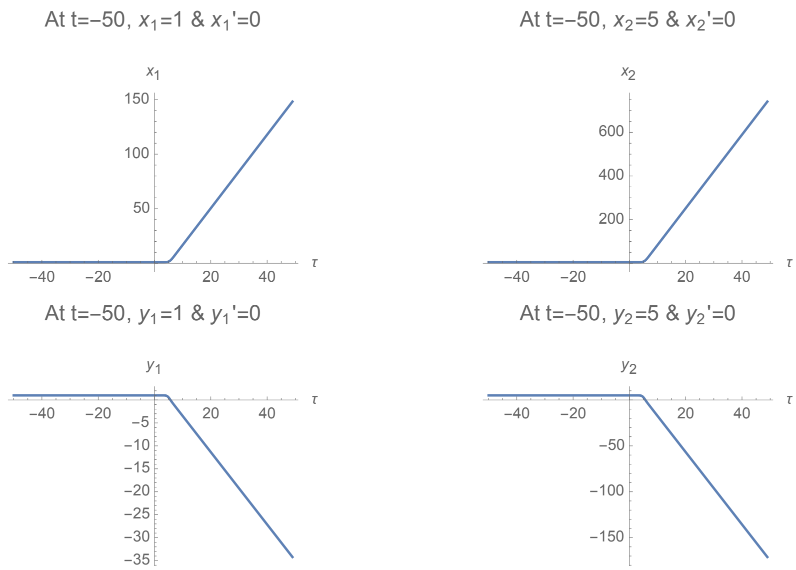

Figure 3.1: Position of the test masses at different times

the corresponding displacement memory is as follows

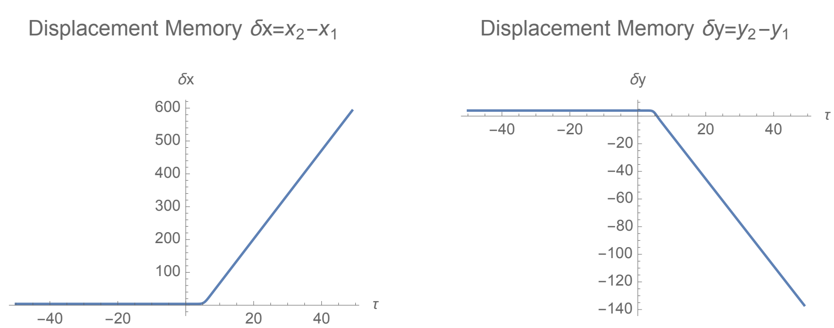

Figure 3.2: Displacement memory between the test masses at different times

Similarly we obtain the velocity of the test masses at different times and its velocity memory shown in figure 3.3 and 3.4 respectively.

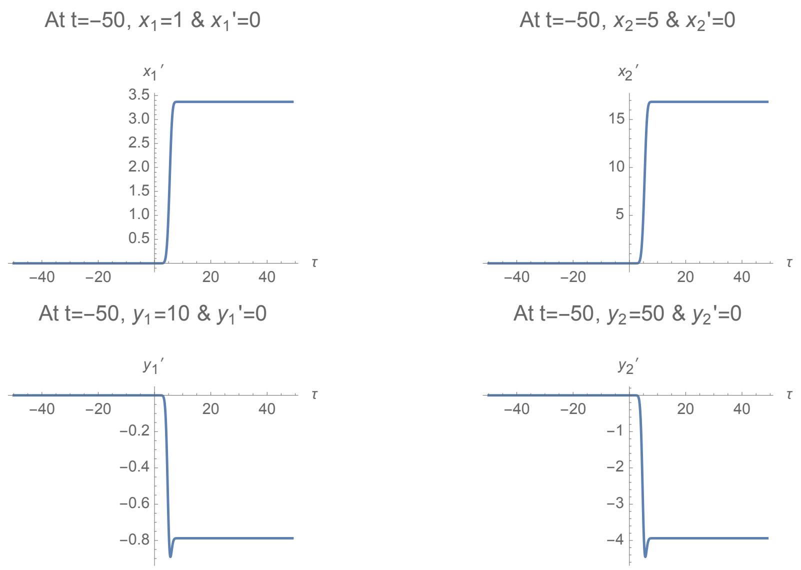

Figure 3.3: Velocity of the test masses at different times

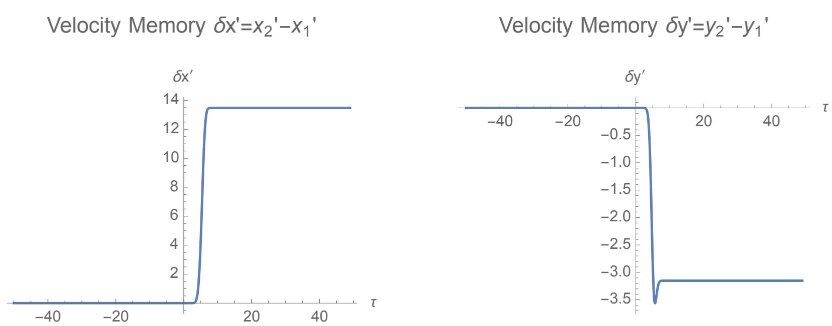

Figure 3.4: Velocity memory between the test masses at different times

Double Barrier Pulse

Now we will be using GW pulse of the form that looks like a double barrier used in previous chapter Figure2.6

$$A_{ij}(u) = \begin{pmatrix} e^{-a(u-5)^2-\frac{b}{(u-5)^2}} & 0 \\ 0 & -e^{-a(u-5)^2-\frac{b}{(u-5)^2}} \end{pmatrix}$$

Solving the geodesic equation (3.7) numerically we obtain the geodesics of the two test masses and its displacement memory shown in figure 3.5 and 3.6 respectively.

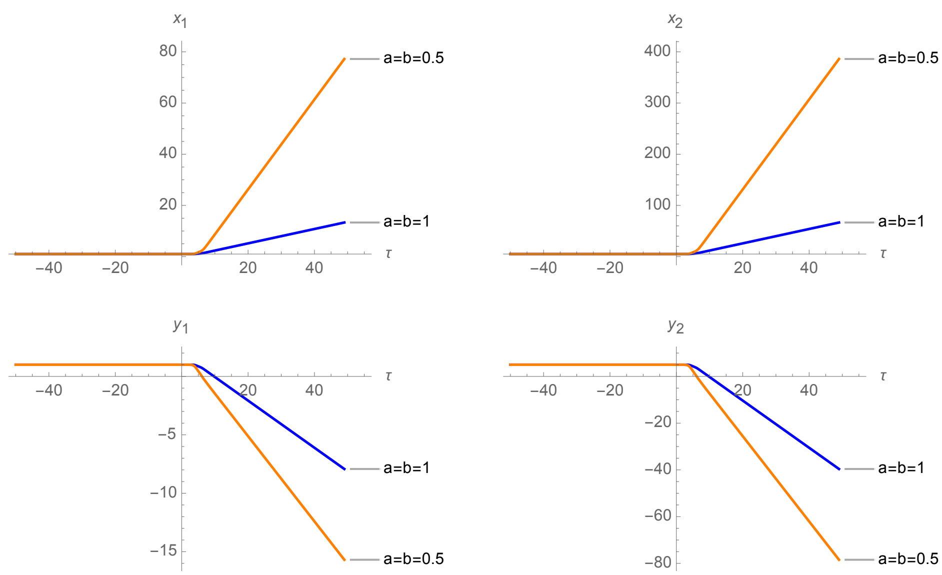

Figure 3.5: Position of the test masses at different times

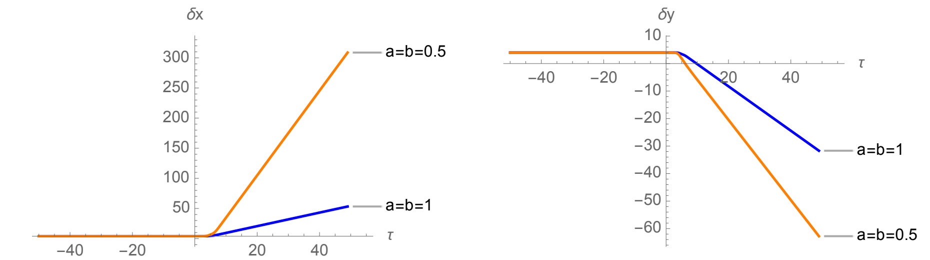

Figure 3.6: Displacement memory between the test masses at different times

As done earlier we obtain the velocity of the test masses at different times and its velocity memory shown in figure 3.7 and 3.8 respectively.

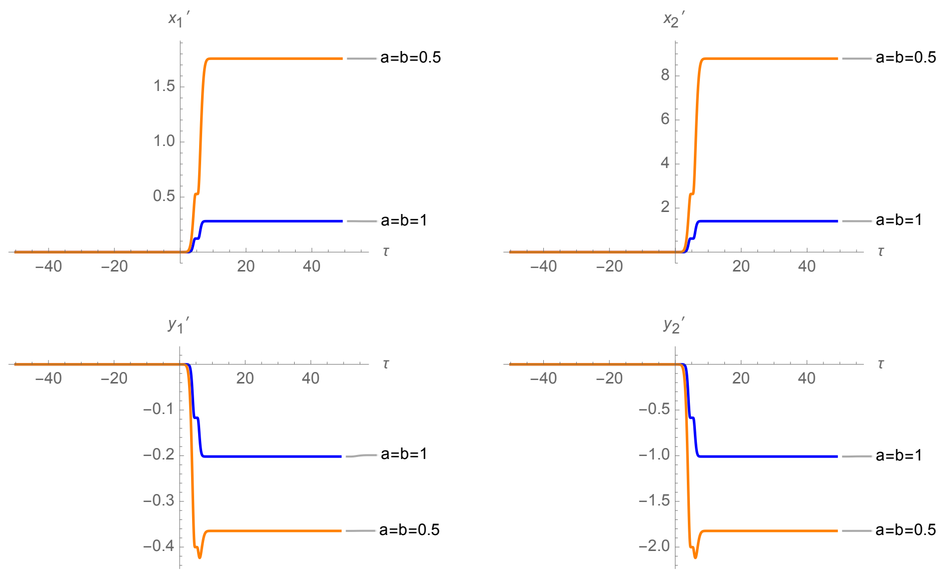

Figure 3.7: Velocity of the test masses at different times

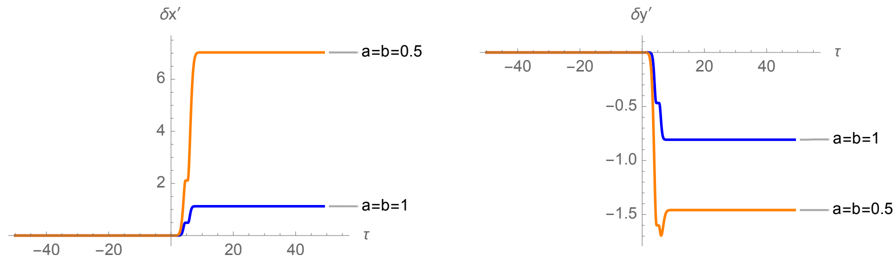

Figure 3.8: Velocity memory between the test masses at different times

3.5 Summary of results

Here also for change in x and y position of the test masses are opposite in nature for gaussian and double barrier pulse. From the displacement memory curve of gaussian or double barrier pulse one can say if x position increases then y position of the mass will montonically decrease with time. In the velocity memory plots of gaussian pulse the velocity becomes constant in single step but for double barrier pulse the velocity changes in two steps i.e. each step for each of the barrier.

Chapter 4

Triangular Pulse in exact radiative spacetime

Here we will analyze the effect of triangular pulse in exact radiative spacetime. We will consider a triangular pulse shown in Figure4.1 which start growing linearly at $u = -\frac{a}{2}$ with slope β and then start decreasing linearly at $u = 0$ with slope $-\beta$ until the pulse reaches zero at $u = \frac{a}{2}$. Before $u = -\frac{a}{2}$ and after $u = \frac{a}{2}$ the profile of the pulse is flat having zero value. Based on the

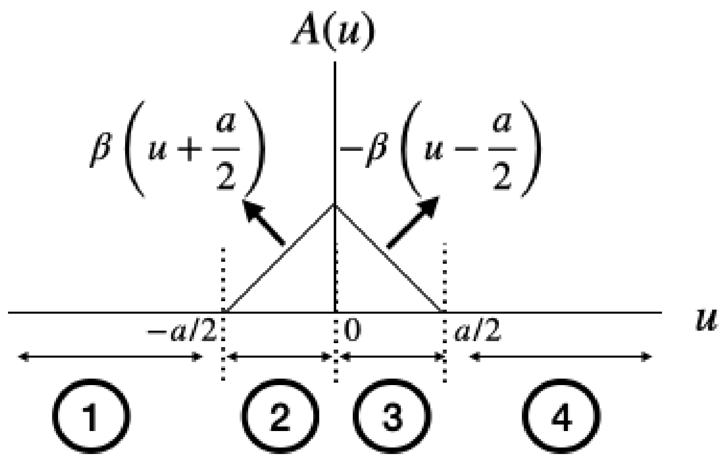

Figure 4.1: Triangular Pulse

piecewise functional expressions of the triangular pulse we divide the pulse into four regions. The region which contain all values of $u \leq -\frac{a}{2}$ belong to Region 1, Region 2 contain $u \in (-\frac{a}{2}, 0]$, Region 3 contain $u \in (0, \frac{a}{2}]$ and $u > \frac{a}{2}$ belong to Region 4.

$$A(u) = \begin{cases} 0 & u \leq -\frac{a}{2} \quad \text{(Region 1)} \\ \beta(u + \frac{a}{2}) & -\frac{a}{2} < u \leq 0 \quad \text{(Region 2)} \\ -\beta(u - \frac{a}{2}) & 0 < u \leq \frac{a}{2} \quad \text{(Region 3)} \\ 0 & u > \frac{a}{2} \quad \text{(Region 4)} \end{cases} \tag{4.1}$$

Since u is a affine parameter, we assume $u = \tau$ and the geodesic equation Eq.(3.7) takes the form

$$\frac{d^2x^i}{du^2} = A^i_j(u)x^j(u) \tag{4.2}$$

4.1 Plus Polarised

Let's take the GW pulse $A_{ij}(u)$ to be triangular in shape and plus polarised in nature. In the above Figure4.1 we have shown our chosen pulse of width a and slope β. Solving Eq.(4.2) analytically, the expressions of $x(u)$ and $y(u)$ are as follows

$$x(u) = \begin{cases} v^x_{\text{init}}\left(u - u_{\text{init}}\right) + x_{\text{init}} & u \leq -\frac{a}{2} \\ C_1 Ai\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] + C_2 Bi\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] & -\frac{a}{2} < u \leq 0 \\ C_3 Ai\left[(-\beta)^{1/3}\left(u - \frac{a}{2}\right)\right] + C_4 Bi\left[(-\beta)^{1/3}\left(u - \frac{a}{2}\right)\right] & 0 < u \leq \frac{a}{2} \\ v^x_{\text{final}}\left(u - \frac{a}{2}\right) + x_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.3}$$

$$y(u) = \begin{cases} v^y_{\text{init}}\left(u - u_{\text{init}}\right) + y_{\text{init}} & u \leq -\frac{a}{2} \\ D_1 Ai\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] + D_2 Bi\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] & -\frac{a}{2} < u \leq 0 \\ D_3 Ai\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] + D_4 Bi\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] & 0 < u \leq \frac{a}{2} \\ v^y_{\text{final}}\left(u - \frac{a}{2}\right) + y_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.4}$$

where $(x_{\text{init}}, y_{\text{init}})$ and $(v^x_{\text{init}}, v^y_{\text{init}})$ are initial position and velocity of the test mass at $u = u_{\text{init}}$. The expressions of the arbitrary constants C_i's and D_i's can be found in Appendix: B [7].

Now, let's consider 2 test masses with initial separation $(\Delta x_{\text{init}}, \Delta y_{\text{init}})$ and relative velocity $(\Delta v^x_{\text{init}}, \Delta v^y_{\text{init}})$. Then, the evolution of displacement memory $(\delta x(u), \delta y(u))$ is given by

$$\delta x(u) = \begin{cases} \Delta v^x_{\text{init}}\left(u - u_{\text{init}}\right) + \Delta x_{\text{init}} & u \leq -\frac{a}{2} \\ \tilde{C}_1 Ai\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] + \tilde{C}_2 Bi\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] & -\frac{a}{2} < u \leq 0 \\ \tilde{C}_3 Ai\left[(-\beta)^{1/3}\left(u - \frac{a}{2}\right)\right] + \tilde{C}_4 Bi\left[(-\beta)^{1/3}\left(u - \frac{a}{2}\right)\right] & 0 < u \leq \frac{a}{2} \\ \Delta v^x_{\text{final}}\left(u - \frac{a}{2}\right) + \Delta x_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.5}$$

$$\delta y(u) = \begin{cases} \Delta v^y_{\text{init}}\left(u - u_{\text{init}}\right) + \Delta y_{\text{init}} & u \leq -\frac{a}{2} \\ \tilde{D}_1 Ai\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] + \tilde{D}_2 Bi\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] & -\frac{a}{2} < u \leq 0 \\ \tilde{D}_3 Ai\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] + \tilde{D}_4 Bi\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] & 0 < u \leq \frac{a}{2} \\ \Delta v^y_{\text{final}}\left(u - \frac{a}{2}\right) + \Delta y_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.6}$$

where $\tilde{C}_i$'s and $\tilde{D}_i$'s are given by

$$\begin{aligned} \tilde{C}_1 &= \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[3]{\beta}} \\ \tilde{C}_2 &= \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}} \end{aligned}$$

$$\begin{aligned} \tilde{C}_3 = &\left(\frac{Bi[\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - (-1)^{1/3}Bi[-\frac{a}{2}(-\beta)^{1/3}]Bi'[\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]}\right) \\ &\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}}\right) \\ &+ \left(\frac{Ai[\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - (-1)^{1/3}Ai'[\frac{a}{2}\beta^{1/3}]Bi[-\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]}\right) \\ &\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[3]{\beta}}\right) \end{aligned}$$

$$\tilde{C}_4 = \left(\frac{-Bi[\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}] + (-1)^{1/3}Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]} \right)$$
$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}} \right)$$
$$+ \left(\frac{-Ai[\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}] + (-1)^{1/3}Ai'[\frac{a}{2}\beta^{1/3}]Ai[-\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]} \right)$$
$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^x{}_{\text{init}} + \Delta x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^x{}_{\text{init}}}{2\sqrt[3]{\beta}} \right)$$

$$\tilde{D}_1 = \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^y{}_{\text{init}} + \Delta y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^y{}_{\text{init}}}{2\sqrt[3]{-\beta}}$$
$$\tilde{D}_2 = \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^y{}_{\text{init}} + \Delta y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^y{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}}$$

$$\tilde{D}_3 = \left(\frac{Bi[\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - (-1)^{1/3}Bi[-\frac{a}{2}\beta^{1/3}]Bi'[\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]} \right)$$
$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^y{}_{\text{init}} + \Delta y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^y{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}} \right)$$
$$+ \left(\frac{Ai[\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - (-1)^{1/3}Ai'[\frac{a}{2}(-\beta)^{1/3}]Bi[-\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]} \right)$$
$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^y{}_{\text{init}} + \Delta y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^y{}_{\text{init}}}{2\sqrt[3]{-\beta}} \right)$$

$$\tilde{D}_4 = \left(\frac{-Bi[\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}] + (-1)^{1/3}Ai[-\frac{a}{2}\beta^{1/3}]Bi'[\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]} \right)$$

$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^{y}{}_{\text{init}} + \Delta y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)\Delta v^{y}{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}} \right)$$

$$+ \left(\frac{-Ai[\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}] + (-1)^{1/3}Ai'[\frac{a}{2}(-\beta)^{1/3}]Ai[-\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]} \right)$$

$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)\Delta v^{y}{}_{\text{init}} + \Delta y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\Delta v^{y}{}_{\text{init}}}{2\sqrt[3]{-\beta}} \right)$$

and

$$\begin{aligned}
\Delta x_{\text{final}} &= \tilde{C}_3 Ai(0) + \tilde{C}_4 Bi(0) \\
\Delta v^{x}_{\text{final}} &= (-\beta)^{1/3}\left(\tilde{C}_3 Ai'(0) + \tilde{C}_4 Bi'(0)\right) \\
\Delta y_{\text{final}} &= \tilde{D}_3 Ai(0) + \tilde{D}_4 Bi(0) \\
\Delta v^{y}_{\text{final}} &= \beta^{1/3}\left(\tilde{D}_3 Ai'(0) + \tilde{D}_4 Bi'(0)\right)
\end{aligned}$$

In the Eq.(4.5) and Eq.(4.6) the displacement memory does not depend on the position or velocity of each test masses. Instead, it is a function of their relative differences which is consistent with the homogeneity of space.

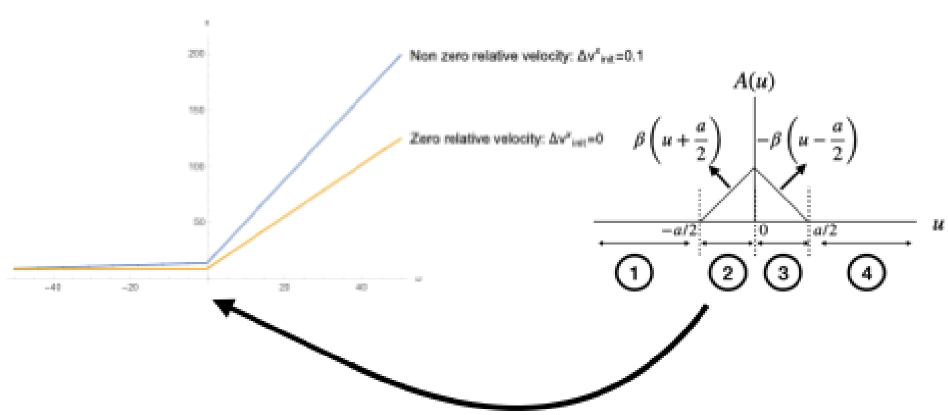

Figure 4.2: Displacement Memory of the test masses along x axis

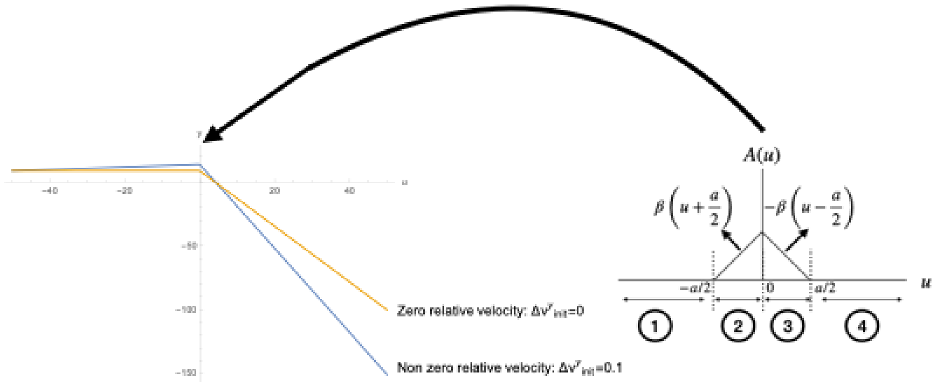

Figure 4.3: Displacement Memory of the test masses along y axis

Displacement memory of two test masses separated by 9 unit along both axis initially with zero initial relative velocity and non-zero initial relative velocity ($\Delta v_{\text{init}} = 0.1$) have been shown in the above two plots where we have taken $a = 1$, $\beta = 1$ and $u_{\text{init}} = -50$.

Now, let's look at the displacement memory graphs for different values of a and β with initial separation of 9 units and zero initial relative velocity at initial time $u_{\text{init}} = -50$.

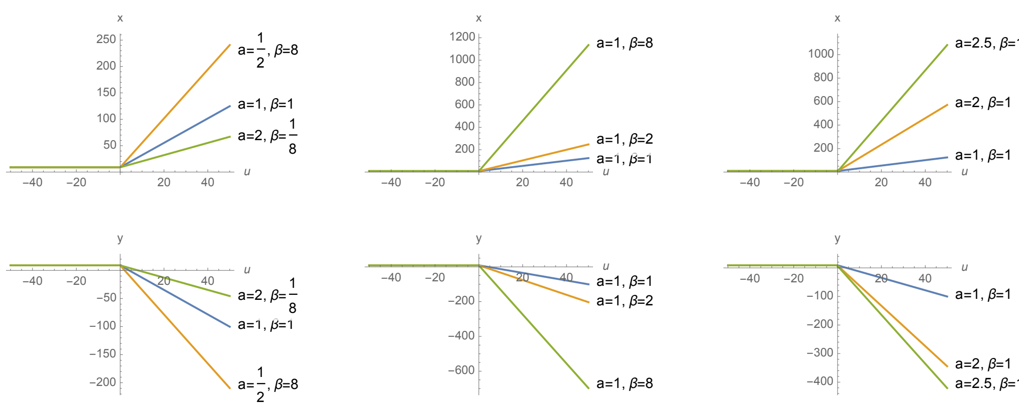

Figure 4.4: Displacement Memory for different values of parameter a and β

The effect of GW on the shape of a ring of particles is an old idea to detect

GW. Here, we will see the effect of this plus polarised metric containing triangular pulse at different times on a set of particles placed circularly with no initial velocity at $u_{\text{init}} = -50$. Displacement memory of these ring of particles w.r.t origin have been shown in Figure4.5 for a triangular pulse of $a = 1$ and $\beta = 1$.

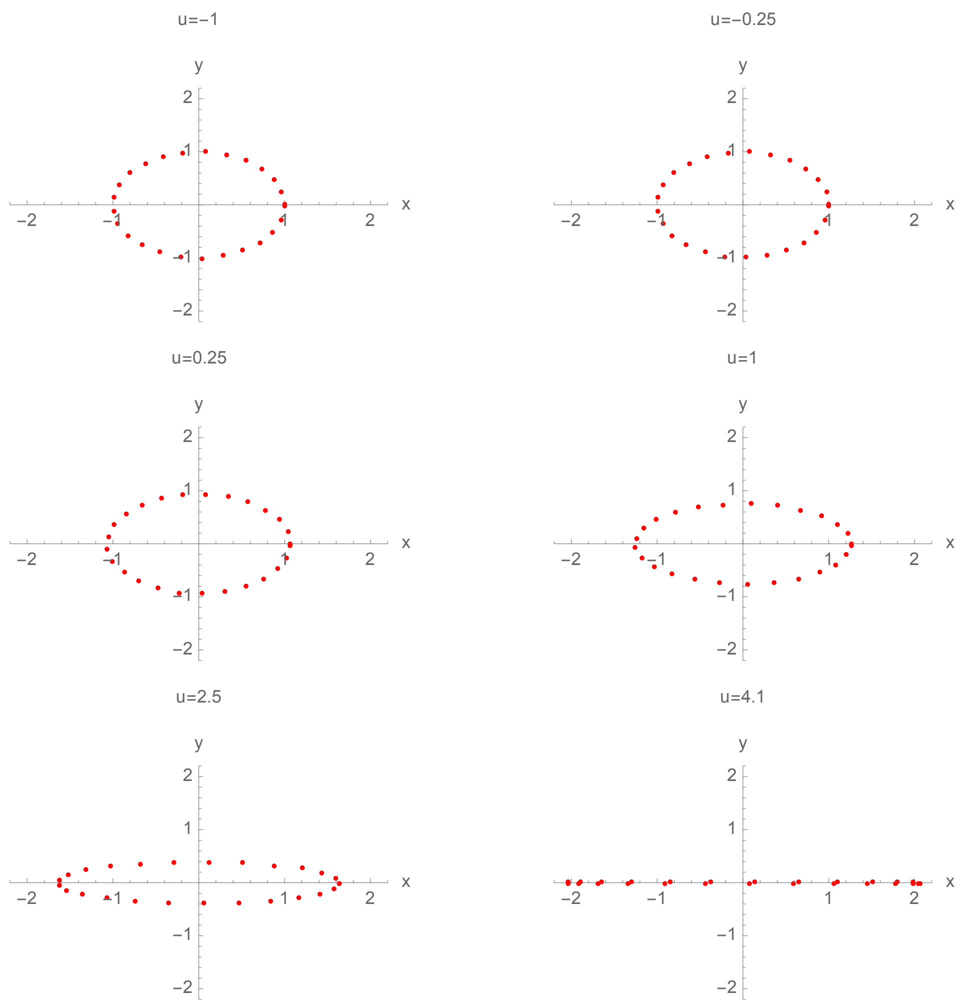

Figure 4.5: Variation of the shape of a ring of particles in presence of a plus polarised triangular pulse

4.2 Cross Polarised

Now, we will take triangular GW pulse in cross polarised metric $A_{ij}(u)$. Solving Eq.(4.2) analytically, the expressions of $x(u)$ and $y(u)$ are as follows

$$x(u) = \begin{cases} v^x_{\text{init}}\,(u - u_{\text{init}}) + x_{\text{init}} & u \leq -\frac{a}{2} \\ \frac{1}{2}(E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + E_2 Bi[\beta^{1/3}(u+\frac{a}{2})] & \\ +E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] + E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})]) & -\frac{a}{2} < u \leq 0 \\ \frac{1}{2}(E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] & \\ +E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] + E_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) & 0 < u \leq \frac{a}{2} \\ v^x_{\text{final}}\left(u - \frac{a}{2}\right) + x_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.7}$$

$$y(u) = \begin{cases} v^y_{\text{init}}\,(u - u_{\text{init}}) + y_{\text{init}} & u \leq -\frac{a}{2} \\ \frac{1}{2}(E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + E_2 Bi[\beta^{1/3}(u+\frac{a}{2})] & \\ -E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] - E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})]) & -\frac{a}{2} < u \leq 0 \\ \frac{1}{2}(E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] & \\ -E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] - E_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) & 0 < u \leq \frac{a}{2} \\ v^y_{\text{final}}\left(u - \frac{a}{2}\right) + y_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.8}$$

where $(x_{\text{init}}, y_{\text{init}})$ and $(v^x_{\text{init}}, v^y_{\text{init}})$ are initial position and velocity of the test mass at $u = u_{\text{init}}$. The expressions of the arbitrary constants E_i's can be found in Appendix: B [7].

Now, let's consider 2 test masses with initial separation $(\Delta x_{\text{init}}, \Delta y_{\text{init}})$ and relative velocity $(\Delta v^x_{\text{init}}, \Delta v^y_{\text{init}})$. Then, the evolution of displacement memory $(\delta x(u), \delta y(u))$ is given by

$$\delta x(u) = \begin{cases} \Delta v^x_{\text{init}}\,(u - u_{\text{init}}) + \Delta x_{\text{init}} & u \leq -\frac{a}{2} \\ \frac{1}{2}(\tilde{E}_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + \tilde{E}_2 Bi[\beta^{1/3}(u+\frac{a}{2})] & \\ +\tilde{E}_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] + \tilde{E}_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})]) & -\frac{a}{2} < u \leq 0 \\ \frac{1}{2}(\tilde{E}_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + \tilde{E}_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] & \\ +\tilde{E}_7 Ai[\beta^{1/3}(u-\frac{a}{2})] + \tilde{E}_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) & 0 < u \leq \frac{a}{2} \\ \Delta v^x_{\text{final}}\left(u - \frac{a}{2}\right) + \Delta x_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.9}$$

$$\delta y(u) = \begin{cases} \Delta v^y_{\text{init}} \left(u - u_{\text{init}}\right) + \Delta y_{\text{init}} & u \leq -\frac{a}{2} \\ \frac{1}{2}(\tilde{E}_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + \tilde{E}_2 Bi[\beta^{1/3}(u+\frac{a}{2})] & \\ -\tilde{E}_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] - \tilde{E}_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})]) & -\frac{a}{2} < u \leq 0 \\ \frac{1}{2}(\tilde{E}_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + \tilde{E}_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] & \\ -\tilde{E}_7 Ai[\beta^{1/3}(u-\frac{a}{2})] - \tilde{E}_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) & 0 < u \leq \frac{a}{2} \\ \Delta v^y_{\text{final}} \left(u - \frac{a}{2}\right) + \Delta y_{\text{final}} & u > \frac{a}{2} \end{cases} \tag{4.10}$$

where $\tilde{E}_i$'s can be found by taking the differences between the E_i for the two test masses. E_i's are functions of $x_{\text{init}}, y_{\text{init}}, v^x_{\text{init}}, v^y_{\text{init}}$ and if we replace these by $\Delta x_{\text{init}}, \Delta y_{\text{init}}, \Delta v^x_{\text{init}}, \Delta v^y_{\text{init}}$ we get $\tilde{E}_i$ from E_i.

Here, we have shown the displacement memory of two test masses separated by 9 unit along both axis initially with zero initial relative velocity and non-zero initial relative velocity ($\Delta v_{\text{init}} = 0.1$) in the following two plots where we have taken $a = 1$, $\beta = 1$ and $u_{\text{init}} = -50$.

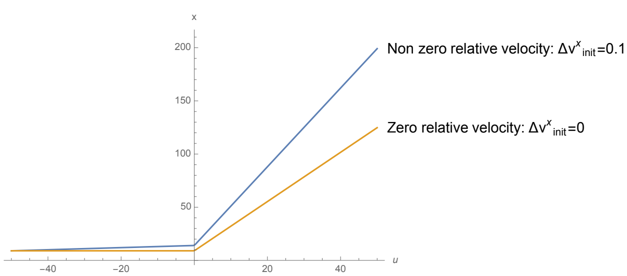

Figure 4.6: Displacement Memory of the test masses along x axis

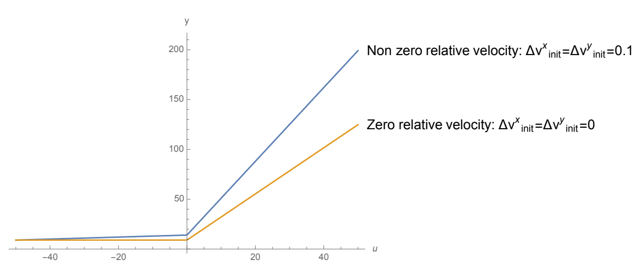

Figure 4.7: Displacement Memory of the test masses along y axis

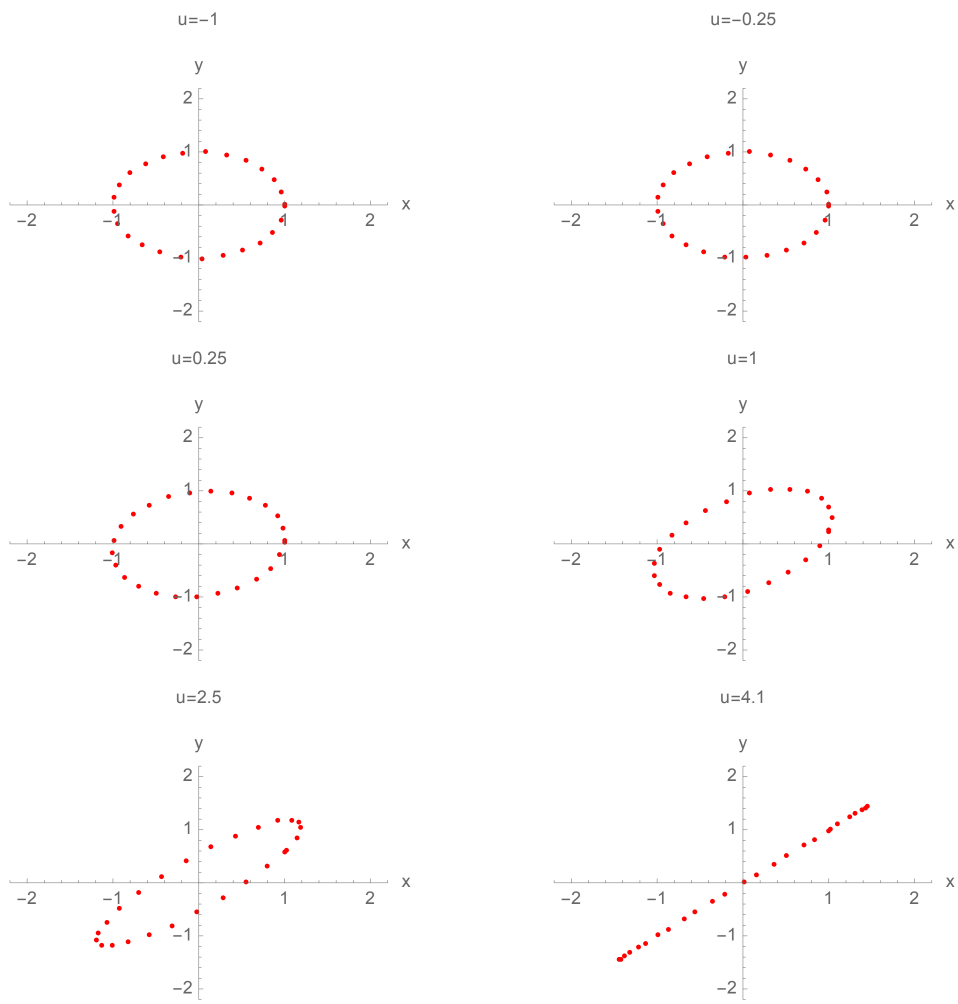

Figure 4.8: Variation of the shape of a ring of particles in presence of a cross polarised triangular pulse

Chapter 5

Conclusions

In this report I have tried to give an overview of the GW memory and persistent memory observable which will set the foundation for my next work. The behaviour of some of the persistent observables in presence of some particular GW have been discussed to a great extent. We have discussed about the analytical form of the displacement memory in presence of a triangular and square pulse and then using this expressions we have analysed the change in the shape of a ring of particles for triangular pulse. Then we have calculated the Jacobi propagators for a square pulse in exact radiative spacetime and compared this result with the analytical solution. Such examples may intrigue readers to think about some experimental setup to detect GW memory. One might think of a setup where two initially separated masses comes closer and collide. Or, some setup where two misaligned masses (ex-laser-LDR pair) become aligned because of GW memory. But one cannot predict about the GW pulse profile so, more than two test masses instead of two in the setup might help in detecting memory.

Chapter 6

Appendix A

6.1 Different types of memory effects and persistent observables: a brief summary

Based on different physical scenarios one can define different types of memory observables for different physical quantities. Out of many such observables displacement memory, persistent relative velocity memory, persistent relative proper time and persistent Lorentz transformation observable are some important memory observables [2]. If two test masses follow geodesics γ and $\bar{\gamma}$ and have a initial and final separation vector ζ^a and $\zeta^{a'}$ at proper times τ_0 and τ_1 respectively of an observer on γ, then the following quantities can be expressed in terms of Riemann tensor as

Displacement Memory:

$$\Delta\zeta^\alpha = -\int_{\tau_0}^{\tau_1} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_3 R^\alpha_{\beta\gamma\delta}(\tau_3)\dot{\gamma}^\beta\zeta^\gamma\dot{\gamma}^\delta$$

This is nothing but of change in geodesic separation in presence of a gravitational wave.

Persistent Relative Velocity Observable:

$$\Delta\dot{\zeta}^\alpha = -\int_{\tau_0}^{\tau_1} d\tau_2 R^\alpha_{\beta\gamma\delta}(\tau_2)\dot{\gamma}^\beta\zeta^\gamma\dot{\gamma}^\delta$$

This gives change in relative velocity of two test masses along these geodesics.

Persistent Relative Proper Time Observable: If we measure the initial separation ζ^α at proper time τ_0 of two observers' clock on γ and $\bar{\gamma}$, the time at the clocks will not be the same while measuring the final separation. Then the difference in proper time will be

$$\Delta\tau = \frac{1}{2}\int_{\tau_0}^{\tau_1} d\tau_2 R_{\alpha\beta\gamma\delta}(\tau_2)\zeta^\alpha\dot{\gamma}^\beta\zeta^\gamma\dot{\gamma}^\delta$$

Persistent Lorentz Transformation Observable: On the geodesics tetrads can be parallel transported along the worldlines and these tetrads are related to each other by Lorentz transformation matrix which shows variation when GW passes by. The transformation matrix can be written as

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \Delta\Omega^\mu_\nu$$

where

$$\Delta\Omega^\mu_\nu = \int_{\tau_0}^{\tau^1} d\tau_2 R^\mu_{\nu\alpha\beta}(\tau_2)\zeta^\alpha\dot{\gamma}^\beta$$

Although we have mentioned about persistent relative proper time and Lorentz transformation observable, we have not used these in this report. We will include these two observable in our future work.

6.2 Geodesics in terms of Jacobi Propagators

The solutions of Eq. (3.7) can be expressed in terms of transverse Jacobi propagators $K^i_j(u', u)$ and $H^i_j(u', u)$

$$x^i(\tau') = K^i_j(u', u)x^j(\tau) + (\tau' - \tau)H^i_j(u', u)\dot{x}^j(\tau) \tag{6.1}$$

which are two point functions of u and u' and satisfy the following differential equations

$$\begin{aligned} \partial^2_{u'}K^i_j(u', u) &= A^i_k(u')K^k_j(u', u) && (6.2)\\ \partial^2_{u'}[(u' - u)H^i_j(u', u)] &= (u' - u)A^i_k(u')H^k_j(u', u) && (6.3) \end{aligned}$$

with boundary conditions

$$\begin{aligned} K^i_j(u, u) &= H^i_j(u, u) = \delta^i_j && (6.4)\\ \partial_{u'}K^i_j(u', u)|_{u'=u} &= \partial_{u'}H^i_j(u', u)|_{u'=u} = 0 && (6.5) \end{aligned}$$

These tranverse Jacobi propagators can be written as

$$K^i_j(u',u) = \sum_{n=0}^{\infty} {}^{(n)}K^i_j(u',u) \tag{6.6}$$

$$H^i_j(u',u) = \sum_{n=0}^{\infty} {}^{(n)}H^i_j(u',u) \tag{6.7}$$

The zeroth order terms are obtained from the boundary conditions given in Eq. (6.4)

$${}^{(0)}K^i_j(u',u) ={}^{(0)} H^i_j(u',u) = \delta^i_j \tag{6.8}$$

At first order, the propagators are calculated using

$${}^{(1)}K^i_j(u',u) = \int_u^{u'} du'' \int_u^{u''} du''' A^i_j(u''') \tag{6.9}$$

$${}^{(1)}H^i_j(u',u) = \int_u^{u'} du'' \int_u^{u''} du''' \frac{u'''-u}{u'-u} A^i_j(u''') \tag{6.10}$$

In the next section we will calculate Jacobi propagators of square GW pulse upto first order and will compare the memory observable obtained from Eq. (6.1) with it's analytical solution.

6.3 Jacobi Propagators for Square Pulse in Plus polarised metric

Now, let's take a square pulse of width a and height H shown in Figure6.2 to study the memory effect in the spacetime. This pulse is sandwiched between $u = -\frac{a}{2}$ and $u = \frac{a}{2}$ and it increases to H from zero and at $u = -\frac{a}{2}$ and again drops to zero at $u = \frac{a}{2}$. Due to piecewise functional definition of the square we divide the whole region into 3 parts. Region 1 belong to $u \leq -\frac{a}{2}$, Region 2 consists of $u \in (-\frac{a}{2}, \frac{a}{2}]$ and Region 3 belong to $u > \frac{a}{2}$.

$$A(u) = \begin{cases} 0 & u \leq -\frac{a}{2} \quad \text{(Region 1)} \\ H & -\frac{a}{2} < u \leq \frac{a}{2} \quad \text{(Region 2)} \\ 0 & u > \frac{a}{2} \quad \text{(Region 4)} \end{cases} \tag{6.11}$$

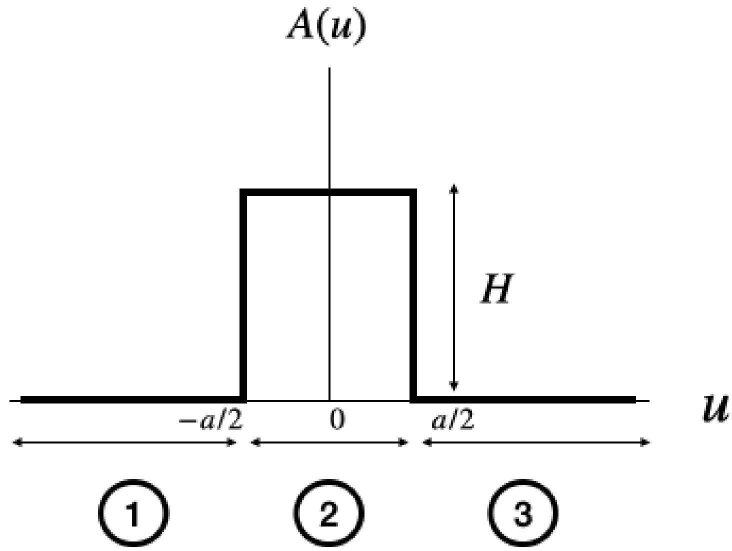

Figure 6.1: Square Pulse

To calculate Jacobi propagators using Eq.(6.9) and Eq.(6.10) it will be easier if we represent the pulse in terms of step function $\mathcal{U}(u)$.

$$A(u) = H\left[\mathcal{U}(u+\frac{a}{2}) - \mathcal{U}(u-\frac{a}{2})\right] \tag{6.12}$$

Before we delve into the calculation of Jacobi propagators, we will calculate some integrals of step function which will come handy in our further calculations.

$$\int_u^{u''} \mathcal{U}(u'''-a)du''' = (u''-a)\mathcal{U}(u''-a) - (u-a)\mathcal{U}(u-a)$$

$$\begin{aligned}\int_u^{u'} (u''-a)\mathcal{U}(u''-a)du'' &= \left[\frac{1}{2}(u''-a)^2\mathcal{U}(u''-a)\right]_u^{u'} - \int_u^{u'} \delta(u''-a)\frac{1}{2}(u''-a)^2du'' \\ &= \frac{1}{2}\left[(u'-a)^2\mathcal{U}(u'-a) - (u-a)^2\mathcal{U}(u-a)\right]\end{aligned}$$

$$\begin{aligned}\int_u^{u'} (u''-a)^2\mathcal{U}(u''-a)du'' &= \left[\frac{1}{3}(u''-a)^3\mathcal{U}(u''-a)\right]_u^{u'} - \int_u^{u'} \delta(u''-a)\frac{1}{3}(u''-a)^3du'' \\ &= \frac{1}{3}\left[(u'-a)^3\mathcal{U}(u'-a) - (u-a)^3\mathcal{U}(u-a)\right]\end{aligned}$$

$$
\begin{aligned}
\int_u^{u'} du'' \int_u^{u''} du''' \frac{u'''-u}{u'-u}\mathcal{U}(u'''-a) &= \int_u^{u'} du'' \int_u^{u''} du''' \left[\frac{u'''-a}{u'-u}\mathcal{U}(u'''-a) - \frac{u-a}{u'-u}\mathcal{U}(u'''-a)\right] \\
&= \int_u^{u'} du'' \frac{1}{u'-u}[\frac{1}{2}(u''-a)^2\mathcal{U}(u''-a) - \frac{1}{2}(u-a)^2\mathcal{U}(u-a) \\
&\quad -(u-a)(u''-a)\mathcal{U}(u''-a) + (u-a)^2\mathcal{U}(u-a)] \\
&= \int_u^{u'} du'' \frac{1}{u'-u}[\frac{1}{2}(u''-a)^2\mathcal{U}(u''-a) \\
&\quad -(u-a)(u''-a)\mathcal{U}(u''-a) + \frac{1}{2}(u-a)^2\mathcal{U}(u-a)] \\
&= \frac{1}{6}\frac{1}{(u'-u)}[(u'-a)^3\mathcal{U}(u'-a) - (u-a)^3\mathcal{U}(u-a)] \\
&\quad -\frac{1}{2}\frac{u-a}{u'-u}[(u'-a)^2\mathcal{U}(u'-a) - (u-a)^2\mathcal{U}(u-a)] \\
&\quad +\frac{1}{2}(u-a)^2\mathcal{U}(u-a) \\
&= \frac{1}{6}\frac{1}{(u'-u)}(u'-a)^3\mathcal{U}(u'-a) - \frac{1}{2}\frac{u-a}{u'-u}(u'-a)^2\mathcal{U}(u'-a) \\
&\quad +\frac{1}{3}\frac{1}{(u'-u)}(u-a)^3\mathcal{U}(u-a) + \frac{1}{2}(u-a)^2\mathcal{U}(u-a)
\end{aligned}
$$

Now, we will proceed to calculate the first order Jacobi propagators for the square pulse of height H in plus polarised metric

$$
\begin{aligned}
A_{11}(u) &= A(u) \\
A_{22}(u) &= -A(u)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(1)}K_1^1(u',u) &= \int_u^{u'} du'' \int_u^{u''} du''' A_1^1(u''') \\
&= H\int_u^{u'} du'' \int_u^{u''} du''' \left[\mathcal{U}(u'''+\frac{a}{2}) - \mathcal{U}(u'''-\frac{a}{2})\right] \\
&= H\int_u^{u'} du''[(u''+\frac{a}{2})\mathcal{U}(u''+\frac{a}{2}) - (u+\frac{a}{2})\mathcal{U}(u+\frac{a}{2}) \\
&\quad -(u''-\frac{a}{2})\mathcal{U}(u''-\frac{a}{2}) + (u-\frac{a}{2})\mathcal{U}(u-\frac{a}{2})]
\end{aligned}
$$

$$
\begin{aligned}
&= H(\frac{1}{2}\left[(u'+\frac{a}{2})^2\mathcal{U}(u'+\frac{a}{2})-(u+\frac{a}{2})^2\mathcal{U}(u+\frac{a}{2})\right] \\
&\quad -\frac{1}{2}\left[(u'-\frac{a}{2})^2\mathcal{U}(u'-\frac{a}{2})-(u-\frac{a}{2})^2\mathcal{U}(u-\frac{a}{2})\right] \\
&\quad -(u'-u)\left[(u+\frac{a}{2})\mathcal{U}(u+\frac{a}{2})-(u-\frac{a}{2})\mathcal{U}(u-\frac{a}{2})\right])
\end{aligned}
$$

$$
{}^{(1)}K_2^2(u',u) = -{}^{(1)}K_1^1(u',u)
$$

$$
\begin{aligned}
{}^{(1)}H_1^1(u',u) &= \int_u^{u'} du'' \int_u^{u''} du''' \frac{u'''-u}{u'-u} A_1^1(u''') \\
&= H \int_u^{u'} du'' \int_u^{u''} du''' \frac{u'''-u}{u'-u}\left[\mathcal{U}(u'''+\frac{a}{2})-\mathcal{U}(u'''-\frac{a}{2})\right] \\
&= H[\frac{1}{6}\frac{1}{(u'-u)}(u'+\frac{a}{2})^3\mathcal{U}(u'+\frac{a}{2})-\frac{1}{2}\frac{u+\frac{a}{2}}{u'-u}(u'+\frac{a}{2})^2\mathcal{U}(u'+\frac{a}{2}) \\
&\quad +\frac{1}{3}\frac{1}{(u'-u)}(u+\frac{a}{2})^3\mathcal{U}(u+\frac{a}{2})+\frac{1}{2}(u+\frac{a}{2})^2\mathcal{U}(u+\frac{a}{2}) \\
&\quad -\frac{1}{6}\frac{1}{(u'-u)}(u'-\frac{a}{2})^3\mathcal{U}(u'-\frac{a}{2})+\frac{1}{2}\frac{u-\frac{a}{2}}{u'-u}(u'-\frac{a}{2})^2\mathcal{U}(u'-\frac{a}{2}) \\
&\quad -\frac{1}{3}\frac{1}{(u'-u)}(u-\frac{a}{2})^3\mathcal{U}(u-\frac{a}{2})-\frac{1}{2}(u-\frac{a}{2})^2\mathcal{U}(u-\frac{a}{2})]
\end{aligned}
$$

$$
{}^{(1)}H_2^2(u',u) = -{}^{(1)}H_1^1(u',u)
$$

Assuming $u=\tau$ the Eq.(6.1) takes the form

$$
x^i(u) = K_j^i(u,u_{\rm init})x^j(u_{\rm init}) + (u-u_{\rm init})H_j^i(u,u_{\rm init})\dot{x}^j(u_{\rm init}) \tag{6.13}
$$

Here we consider $u_{\rm init} << -\frac{a}{2}$ belong to Region 1. Now, we will find the expressions of $x(u)$ and $y(u)$ for the three regions of Figure6.2 using the previously calculated Jacobi propagators in Eq.(6.13).

In the Region 1 ($u \leq -\frac{a}{2}$),

$$
\begin{aligned}
x(u) &= K_1^1(u, u_{\text{init}})x(u_{\text{init}}) + (u - u_{\text{init}})H_1^1(u, u_{\text{init}})\dot{x}(u_{\text{init}}) \\
&= [1+0]x_{\text{init}} + (u - u_{\text{init}})[1+0]v_{\text{init}}^x \\
&= v_{\text{init}}^x(u - u_{\text{init}}) + x_{\text{init}}
\end{aligned}
$$

$$
\begin{aligned}
y(u) &= K_2^2(u, u_{\text{init}})y(u_{\text{init}}) + (u - u_{\text{init}})H_2^2(u, u_{\text{init}})\dot{y}(u_{\text{init}}) \\
&= [1+0]y_{\text{init}} + (u - u_{\text{init}})[1+0]v_{\text{init}}^y \\
&= v_{\text{init}}^y(u - u_{\text{init}}) + y_{\text{init}}
\end{aligned}
$$

In the Region 2 ($-\frac{a}{2} < u \leq \frac{a}{2}$),

$$
\begin{aligned}
x(u) &= K_1^1(u, u_{\text{init}})x(u_{\text{init}}) + (u - u_{\text{init}})H_1^1(u, u_{\text{init}})\dot{x}(u_{\text{init}}) \\
&= \left[1 + \frac{H}{2}(u + \frac{a}{2})^2\right] x_{\text{init}} \\
&\quad + (u - u_{\text{init}}) \left[1 + \frac{H}{6(u - u_{\text{init}})}(u + \frac{a}{2})^3 - \frac{H(u_{\text{init}} + \frac{a}{2})}{2(u - u_{\text{init}})}(u + \frac{a}{2})^2\right] v_{\text{init}}^x \\
&= [v_{\text{init}}^x(u - u_{\text{init}}) + x_{\text{init}}] + \frac{H}{2}(u + \frac{a}{2})^2 x_{\text{init}} + v_{\text{init}}^x \left[\frac{H}{6}(u + \frac{a}{2})^3 - \frac{H}{2}(u_{\text{init}} + \frac{a}{2})(u + \frac{a}{2})^2\right]
\end{aligned}
$$

$$
\begin{aligned}
y(u) &= K_2^2(u, u_{\text{init}})y(u_{\text{init}}) + (u - u_{\text{init}})H_2^2(u, u_{\text{init}})\dot{y}(u_{\text{init}}) \\
&= \left[1 - \frac{H}{2}(u + \frac{a}{2})^2\right] y_{\text{init}} \\
&\quad + (u - u_{\text{init}}) \left[1 - \frac{H}{6(u - u_{\text{init}})}(u + \frac{a}{2})^3 + \frac{H(u_{\text{init}} + \frac{a}{2})}{2(u - u_{\text{init}})}(u + \frac{a}{2})^2\right] v_{\text{init}}^y \\
&= [v_{\text{init}}^y(u - u_{\text{init}}) + y_{\text{init}}] - \frac{H}{2}(u + \frac{a}{2})^2 y_{\text{init}} - v_{\text{init}}^y \left[\frac{H}{6}(u + \frac{a}{2})^3 - \frac{H}{2}(u_{\text{init}} + \frac{a}{2})(u + \frac{a}{2})^2\right]
\end{aligned}
$$

In the Region 3 ($u > \frac{a}{2}$)

$$
\begin{aligned}
x(u) &= K_1^1(u, u_{\text{init}})x(u_{\text{init}}) + (u - u_{\text{init}})H_1^1(u, u_{\text{init}})\dot{x}(u_{\text{init}}) \\
&= \left[1 + \frac{H}{2}(u + \frac{a}{2})^2 - \frac{H}{2}(u - \frac{a}{2})^2\right] x_{\text{init}} + (u - u_{\text{init}})[1 + \frac{H}{6(u - u_{\text{init}})}[(u + \frac{a}{2})^3 - (u - \frac{a}{2})^3] \\
&\quad - \frac{H(u_{\text{init}} + \frac{a}{2})}{2(u - u_{\text{init}})}(u + \frac{a}{2})^2 + \frac{H(u_{\text{init}} - \frac{a}{2})}{2(u - u_{\text{init}})}(u - \frac{a}{2})^2] v_{\text{init}}^x
\end{aligned}
$$

$$
\begin{aligned}
&= [v^x_{\text{init}}(u-u_{\text{init}})+x_{\text{init}}]+Huax_{\text{init}}+v^x_{\text{init}}[\frac{H}{6}((u+\frac{a}{2})^3-(u-\frac{a}{2})^3) \\
&\quad -\frac{H}{2}(u_{\text{init}}+\frac{a}{2})(u+\frac{a}{2})^2+\frac{H}{2}(u_{\text{init}}-\frac{a}{2})(u-\frac{a}{2})^2]
\end{aligned}
$$

$$
\begin{aligned}
y(u) &= K^2_2(u,u_{\text{init}})y(u_{\text{init}})+(u-u_{\text{init}})H^2_2(u,u_{\text{init}})\dot{y}(u_{\text{init}}) \\
&= \left[1-\frac{H}{2}(u+\frac{a}{2})^2+\frac{H}{2}(u-\frac{a}{2})^2\right]y_{\text{init}}+(u-u_{\text{init}})[1-\frac{H}{6(u-u_{\text{init}})}[(u+\frac{a}{2})^3-(u-\frac{a}{2})^3] \\
&\quad +\frac{H(u_{\text{init}}+\frac{a}{2})}{2(u-u_{\text{init}})}(u+\frac{a}{2})^2-\frac{H(u_{\text{init}}-\frac{a}{2})}{2(u-u_{\text{init}})}(u-\frac{a}{2})^2]v^y_{\text{init}} \\
&= [v^y_{\text{init}}(u-u_{\text{init}})+y_{\text{init}}]-Huay_{\text{init}}-v^y_{\text{init}}[\frac{H}{6}((u+\frac{a}{2})^3-(u-\frac{a}{2})^3) \\
&\quad -\frac{H}{2}(u_{\text{init}}+\frac{a}{2})(u+\frac{a}{2})^2+\frac{H}{2}(u_{\text{init}}-\frac{a}{2})(u-\frac{a}{2})^2]
\end{aligned}
$$

Now, in the following figure we have shown the comparison of the displacement memory (along x axis) obtained analytically and using Jacobi propagators

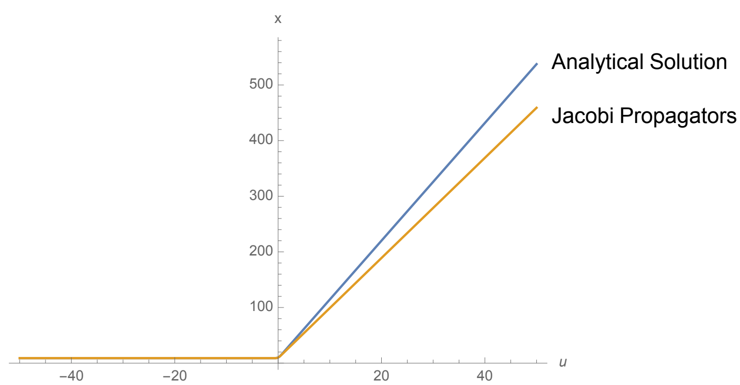

Figure 6.2: Comparison between analytical solution and solution in terms of Jacobi propagators

Chapter 7

Appendix: B

Here we will show complete derivation of analytical solution of Geodesic Eq. (4.2)

7.1 Plus Polarised Triangular Pulse

For the Region 1 in Figure4.1,

$$\begin{aligned}
\frac{d^2x}{du^2} &= 0 \\
x(u) &= v^x_{\text{init}}(u - u_{\text{init}}) + x_{\text{init}} \\
\dot{x}(u) &= v^x_{\text{init}}
\end{aligned}$$

Similarly, for y component it follows

$$\begin{aligned}
\frac{d^2y}{u} &= 0 \\
y(u) &= v^y_{\text{init}}(u - u_{\text{init}}) + y_{\text{init}} \\
\dot{y}(u) &= v^y_{\text{init}}
\end{aligned}$$

where $(x_{\text{init}}, y_{\text{init}})$ and $(v^x_{\text{init}}, v^y_{\text{init}})$ are initial position and velocity of the test mass at $u = u_{\text{init}}$.

In the Region 2,

$$\frac{d^2x}{du^2} = \beta\left(u + \frac{a}{2}\right)x(u)$$

Substituting $z = \beta^{1/3}\left(u + \frac{a}{2}\right)$ we get,

$$\begin{aligned}
\frac{d^2x}{dz^2} &= zx \\
x(u) &= C_1 Ai(z) + C_2 Bi(z) \\
x(u) &= C_1 Ai\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] + C_2 Bi\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] \\
\dot{x}(u) &= \beta^{1/3}\left(C_1 Ai'\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right] + C_2 Bi'\left[\beta^{1/3}\left(u + \frac{a}{2}\right)\right]\right)
\end{aligned}$$

where arbitrary constants C_1 and C_2 are obtained by equating the expression of $x(u)$ and $\dot{x}(u)$ for Region 1 and 2 at their junction $u = -\frac{a}{2}$

$$\begin{bmatrix} Ai(0) & Bi(0) \\ \beta^{1/3}Ai'(0) & \beta^{1/3}Bi'(0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} v^x_{\text{init}}(-\frac{a}{2} - u_{\text{init}}) + x_{\text{init}} \\ v^x_{\text{init}} \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^x{}_{\text{init}} + x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)v^x{}_{\text{init}}}{2\sqrt[3]{\beta}} \\ \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^x{}_{\text{init}} + x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)v^x{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}} \end{bmatrix}$$

And for y coordinate,

$$\frac{d^2y}{du^2} = -\beta\left(u + \frac{a}{2}\right)y(u)$$

Substituting $z = (-\beta)^{1/3}\left(u + \frac{a}{2}\right)$ we get,

$$\begin{aligned}
\frac{d^2y}{dz^2} &= zy \\
y(u) &= D_1 Ai(z) + D_2 Bi(z) \\
y(u) &= D_1 Ai\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] + D_2 Bi\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] \\
\dot{y}(u) &= (-\beta)^{1/3}\left(D_1 Ai'\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right] + D_2 Bi'\left[(-\beta)^{1/3}\left(u + \frac{a}{2}\right)\right]\right)
\end{aligned}$$

where arbitrary constants D_1 and D_2 are obtained in similar way by equating expressions of $y(\tau')$ and $\dot{y}(\tau')$ for Region 1 and 2 at $u' = -\frac{a}{2}$

$$\begin{bmatrix} Ai(0) & Bi(0) \\ (-\beta)^{1/3}Ai'(0) & (-\beta)^{1/3}Bi'(0) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} v^y_{\text{init}}(-\frac{a}{2} - u_{\text{init}}) + y_{\text{init}} \\ v^y_{\text{init}} \end{bmatrix}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)v^y{}_{\text{init}}+y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[3]{-\beta}} \\ \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)v^y{}_{\text{init}}+y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}} \end{bmatrix}$$

In the Region 3,

$$\frac{d^2x}{du^2} = -\beta\left(u-\frac{a}{2}\right)x(u) \tag{7.1}$$

Substituting $z = (-\beta)^{1/3}\left(u-\frac{a}{2}\right)$ we get,

$$\begin{aligned} \frac{d^2x}{dz^2} &= zx \\ x(u) &= C_3Ai(z) + C_4Bi(z) \\ x(u) &= C_3Ai\left[(-\beta)^{1/3}\left(u-\frac{a}{2}\right)\right] + C_4Bi\left[(-\beta)^{1/3}\left(u-\frac{a}{2}\right)\right] \\ \dot{x}(u) &= (-\beta)^{1/3}\left(C_3Ai'\left[(-\beta)^{1/3}\left(u-\frac{a}{2}\right)\right] + C_4Bi'\left[(-\beta)^{1/3}\left(u-\frac{a}{2}\right)\right]\right) \end{aligned}$$

where arbitrary constants C_3 and C_4 can be calculated by equating the expression of $x(u)$ and $\dot{x}(u)$ for Region 2 and 3 at their junction $u = 0$

$$\begin{bmatrix} Ai[-(-\beta)^{1/3}\frac{a}{2}] & Bi[-(-\beta)^{1/3}\frac{a}{2}] \\ (-\beta)^{1/3}Ai'[-(-\beta)^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Bi'[-(-\beta)^{1/3}\frac{a}{2}] \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$$

$$= \begin{bmatrix} Ai[\beta^{1/3}\frac{a}{2}] & Bi[\beta^{1/3}\frac{a}{2}] \\ \beta^{1/3}Ai'[\beta^{1/3}\frac{a}{2}] & \beta^{1/3}Bi'[\beta^{1/3}\frac{a}{2}] \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\begin{aligned} C_3 = &\left(\frac{Bi[\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - (-1)^{1/3}Bi[-\frac{a}{2}(-\beta)^{1/3}]Bi'[\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]}\right) \\ &\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)v^x{}_{\text{init}}+x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)v^x{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}}\right) \\ &+\left(\frac{Ai[\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - (-1)^{1/3}Ai'[\frac{a}{2}\beta^{1/3}]Bi[-\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]}\right) \\ &\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)v^x{}_{\text{init}}+x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)v^x{}_{\text{init}}}{2\sqrt[3]{\beta}}\right) \end{aligned}$$

$$C_4 = \left(\frac{-Bi[\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}] + (-1)^{1/3}Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]} \right)$$

$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right) v^{x}{}_{\text{init}} + x_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right) v^{x}{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{\beta}} \right)$$

$$+ \left(\frac{-Ai[\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}] + (-1)^{1/3}Ai'[\frac{a}{2}\beta^{1/3}]Ai[-\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}(-\beta)^{1/3}] - Bi[-\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}(-\beta)^{1/3}]} \right)$$

$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right) v^{x}{}_{\text{init}} + x_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right) v^{x}{}_{\text{init}}}{2\sqrt[3]{\beta}} \right)$$

Similarly, for y coordinate we get

$$\frac{d^2y}{dz^2} = \beta\left(u - \frac{a}{2}\right) y(u)$$

Substituting $z = \beta^{1/3}\left(u - \frac{a}{2}\right)$ we get,

$$\begin{aligned}
\frac{d^2y}{dz^2} &= zy \\
y(u) &= D_3 Ai(z) + D_4 Bi(z) \\
y(u) &= D_3 Ai\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] + D_4 Bi\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] \\
\dot{y}(u) &= \beta^{1/3}\left(C_3 Ai'\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right] + C_4 Bi'\left[\beta^{1/3}\left(u - \frac{a}{2}\right)\right]\right)
\end{aligned}$$

where arbitrary constants D_3 and D_4 can be calculated using the expression of $y(\tau')$ and $\dot{y}(\tau')$ for Region 2 and 3 at their junction $u' = 0$

$$\begin{bmatrix} Ai[-\beta^{1/3}\frac{a}{2}] & Bi[-\beta^{1/3}\frac{a}{2}] \\ \beta^{1/3}Ai'[-\beta^{1/3}\frac{a}{2}] & \beta^{1/3}Bi'[-\beta^{1/3}\frac{a}{2}] \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix}$$

$$= \begin{bmatrix} Ai[(-\beta)^{1/3}\frac{a}{2}] & Bi[(-\beta)^{1/3}\frac{a}{2}] \\ (-\beta)^{1/3}Ai'[(-\beta)^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Bi'[(-\beta)^{1/3}\frac{a}{2}] \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$D_3 = \left(\frac{Bi[\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - (-1)^{1/3}Bi[-\frac{a}{2}\beta^{1/3}]Bi'[\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]}\right)$$
$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^y{}_{\text{init}} + y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}}\right)$$
$$+\left(\frac{Ai[\frac{a}{2}(-\beta)^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - (-1)^{1/3}Ai'[\frac{a}{2}(-\beta)^{1/3}]Bi[-\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]}\right)$$
$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^y{}_{\text{init}} + y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[3]{-\beta}}\right)$$

$$D_4 = \left(\frac{-Bi[\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}] + (-1)^{1/3}Ai[-\frac{a}{2}\beta^{1/3}]Bi'[\frac{a}{2}(-\beta)^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]}\right)$$
$$\left(\frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^y{}_{\text{init}} + y_{\text{init}}\right) + \frac{\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[6]{3}\sqrt[3]{-\beta}}\right)$$
$$+\left(\frac{-Ai[\frac{a}{2}(-\beta)^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}] + (-1)^{1/3}Ai'[\frac{a}{2}(-\beta)^{1/3}]Ai[-\frac{a}{2}\beta^{1/3}]}{Ai[-\frac{a}{2}\beta^{1/3}]Bi'[-\frac{a}{2}\beta^{1/3}] - Bi[-\frac{a}{2}\beta^{1/3}]Ai'[-\frac{a}{2}\beta^{1/3}]}\right)$$
$$\left(\frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2} - u_{\text{init}}\right)v^y{}_{\text{init}} + y_{\text{init}}\right) - \frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)v^y{}_{\text{init}}}{2\sqrt[3]{-\beta}}\right)$$

In the Region 4,

$$\begin{aligned}
\frac{d^2x}{du^2} &= 0 \\
x(u) &= v^x_{\text{final}}\left(u - \frac{a}{2}\right) + x_{\text{final}} \\
\dot{x}(u) &= v^x_{\text{final}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2y}{du^2} &= 0 \\
y(u) &= v^y_{\text{final}}\left(u - \frac{a}{2}\right) + y_{\text{final}} \\
\dot{y}(u) &= v^y_{\text{final}}
\end{aligned}$$

where (x_{final}, y_{final}) and $(v^x_{final}, v^y_{final})$ are final position and velocity of the test mass respectively at $u = \frac{a}{2}$.

Because of continuity relation of position and velocity vector between Region 3 and 4, at $u' = \frac{a}{2}$ we get

$$\begin{aligned}
x_{\text{final}} &= C_3 Ai(0) + C_4 Bi(0) \\
v^x_{\text{final}} &= (-\beta)^{1/3} \left(C_3 Ai'(0) + C_4 Bi'(0) \right) \\
y_{\text{final}} &= D_3 Ai(0) + D_4 Bi(0) \\
v^y_{\text{final}} &= \beta^{1/3} \left(D_3 Ai'(0) + D_4 Bi'(0) \right)
\end{aligned}$$

7.2 Cross Polarised Triangular Pulse

In the Region 1 of Figure4.1,

$$\begin{aligned}
\frac{d^2x}{du^2} &= 0 \\
x(u) &= v^x_{\text{init}}(u - u_{\text{init}}) + x_{\text{init}} \\
\dot{x}(u) &= v^x_{\text{init}}
\end{aligned}$$

Similarly, for y component it follows

$$\begin{aligned}
\frac{d^2y}{u} &= 0 \\
y(u) &= v^y_{\text{init}}(u - u_{\text{init}}) + y_{\text{init}} \\
\dot{y}(u) &= v^y_{\text{init}}
\end{aligned}$$

where $(x_{\text{init}}, y_{\text{init}})$ and $(v^x_{\text{init}}, v^y_{\text{init}})$ are initial position and velocity of the test mass at $u = u_{\text{init}}$.

In the Region 2,

$$\frac{d^2x}{du^2} = \beta \left(u + \frac{a}{2} \right) y(u) \tag{7.2}$$

and

$$\frac{d^2y}{du^2} = \beta \left(u + \frac{a}{2} \right) x(u) \tag{7.3}$$

We can not solve these coupled differential equations individually. To solve Eq.(7.2) and Eq.(7.3) we need to add and subtract these equations

$$\frac{d^2}{du^2}(x+y) = \beta\left(u+\frac{a}{2}\right)(x+y) \tag{7.4}$$

$$\frac{d^2}{du^2}(x-y) = -\beta\left(u+\frac{a}{2}\right)(x-y) \tag{7.5}$$

From the solution of Eq.(7.1) and Eq.(7.1) in section 7.1 of Appendix: B7 we can write the solutions of the above equations Eq.(7.4) and Eq.(7.5) as

$$x+y = E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + E_2 Bi[\beta^{1/3}(u+\frac{a}{2})] \tag{7.6}$$

$$x-y = E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] + E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})] \tag{7.7}$$

Now adding and subtracting these two equations Eq.(7.6) and Eq.(7.7) we get the expressions of $x(u)$ and $y(u)$ for Region 2

$$x(u) = \frac{1}{2}(E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + E_2 Bi[\beta^{1/3}(u+\frac{a}{2})]$$
$$+E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] + E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})])$$
$$y(u) = \frac{1}{2}(E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + E_2 Bi[\beta^{1/3}(u+\frac{a}{2})]$$
$$-E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] - E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})])$$

and

$$\dot{x}(u) = \frac{1}{2}(\beta^{1/3} E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + \beta^{1/3} E_2 Bi[\beta^{1/3}(u+\frac{a}{2})]$$
$$+(-\beta)^{1/3} E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] + (-\beta)^{1/3} E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})])$$
$$\dot{y}(u) = \frac{1}{2}(\beta^{1/3} E_1 Ai[\beta^{1/3}(u+\frac{a}{2})] + \beta^{1/3} E_2 Bi[\beta^{1/3}(u+\frac{a}{2})]$$
$$-(-\beta)^{1/3} E_3 Ai[(-\beta)^{1/3}(u+\frac{a}{2})] - (-\beta)^{1/3} E_4 Bi[(-\beta)^{1/3}(u+\frac{a}{2})])$$

where arbitrary constants E_i's are obtained by equating expressions of $x(u), y(u), \dot{x}(u)$ and $\dot{y}(u)$ for Region 1 and 2 at $u = -\frac{a}{2}$

$$\frac{1}{2}\begin{bmatrix} Ai(0) & Bi(0) & Ai(0) & Bi(0) \\ Ai(0) & Bi(0) & -Ai(0) & -Bi(0) \\ \beta^{1/3}Ai'(0) & \beta^{1/3}Bi'(0) & (-\beta)^{1/3}Ai'(0) & (-\beta)^{1/3}Bi'(0) \\ \beta^{1/3}Ai'(0) & \beta^{1/3}Bi'(0) & -(-\beta)^{1/3}Ai'(0) & -(-\beta)^{1/3}Bi'(0) \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} v^x_{\text{init}}(-\frac{a}{2} - u_{\text{init}}) + y_{\text{init}} \\ v^y_{\text{init}}(-\frac{a}{2} - u_{\text{init}}) + y_{\text{init}} \\ v^x_{\text{init}} \\ v^y_{\text{init}} \end{bmatrix}$$

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)\left(v^x{}_{\text{init}}+v^y{}_{\text{init}}\right)+x_{\text{init}}+y_{\text{init}}\right)-\frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\left(v^x{}_{\text{init}}+v^y{}_{\text{init}}\right)}{2\sqrt[3]{\beta}} \\ \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)\left(v^x{}_{\text{init}}+v^y{}_{\text{init}}\right)+x_{\text{init}}+y_{\text{init}}\right)+\frac{\Gamma\left(\frac{1}{3}\right)\left(v^x{}_{\text{init}}+v^y{}_{\text{init}}\right)}{2\sqrt[6]{3}\sqrt[3]{\beta}} \\ \frac{1}{2}3^{2/3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)\left(v^x{}_{\text{init}}-v^y{}_{\text{init}}\right)+x_{\text{init}}-y_{\text{init}}\right)-\frac{\sqrt[3]{3}\Gamma\left(\frac{1}{3}\right)\left(v^x{}_{\text{init}}-v^y{}_{\text{init}}\right)}{2\sqrt[3]{-\beta}} \\ \frac{1}{2}\sqrt[6]{3}\Gamma\left(\frac{2}{3}\right)\left(\left(-\frac{a}{2}-u_{\text{init}}\right)\left(v^x{}_{\text{init}}-v^y{}_{\text{init}}\right)+x_{\text{init}}-y_{\text{init}}\right)+\frac{\Gamma\left(\frac{1}{3}\right)\left(v^x{}_{\text{init}}-v^y{}_{\text{init}}\right)}{2\sqrt[6]{3}\sqrt[3]{-\beta}} \end{bmatrix}$$

In the Region 3,

$$\frac{d^2x}{du^2} = -\beta\left(u - \frac{a}{2}\right)y(u) \tag{7.8}$$

and

$$\frac{d^2y}{du^2} = -\beta\left(u - \frac{a}{2}\right)x(u) \tag{7.9}$$

Adding and subtracting Eq.(7.8) and Eq.(7.9) we get

$$\frac{d^2}{du^2}(x+y) = -\beta\left(u - \frac{a}{2}\right)(x+y) \tag{7.10}$$

$$\frac{d^2}{du^2}(x-y) = \beta\left(u - \frac{a}{2}\right)(x-y) \tag{7.11}$$

From the solution of Eq.(7.1) and Eq.(7.1) in section 7.1 of Appendix: B7 we can write the solutions of the above equations Eq.(7.10) and Eq.(7.11) as

$$x+y = E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] \tag{7.12}$$

$$x-y = E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] + E_8 Bi[\beta^{1/3}(u-\frac{a}{2})] \tag{7.13}$$

Now adding and subtracting these two equations Eq.(7.12) and Eq.(7.13) we get the expressions of $x(u)$ and $y(u)$ for Region 3

$$\begin{aligned} x(u) = \frac{1}{2}(&E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] \\ &+E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] + E_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) \\ y(u) = \frac{1}{2}(&E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] \\ &-E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] - E_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) \end{aligned}$$

and

$$
\begin{aligned}
\dot{x}(u) = \frac{1}{2}((-\beta)^{1/3}E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + (-\beta)^{1/3}E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] \\
+\beta^{1/3}E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] + \beta^{1/3}E_8 Bi[\beta^{1/3}(u-\frac{a}{2})]) \\
\dot{y}(u) = \frac{1}{2}((-\beta)^{1/3}E_5 Ai[(-\beta)^{1/3}(u-\frac{a}{2})] + (-\beta)^{1/3}E_6 Bi[(-\beta)^{1/3}(u-\frac{a}{2})] \\
-\beta^{1/3}E_7 Ai[\beta^{1/3}(u-\frac{a}{2})] - \beta^{1/3}E_8 Bi[\beta^{1/3}(u-\frac{a}{2})])
\end{aligned}
$$

where arbitrary constants E_i's are calculated from the expressions of $x(u), y(u), \dot{x}(u)$ and $\dot{y}(u)$ for Region 2 and 3 at $u = 0$

$$
\begin{bmatrix}
Ai[-(-\beta)^{1/3}\frac{a}{2}] & Bi[-(-\beta)^{1/3}\frac{a}{2}] & Ai[-\beta^{1/3}\frac{a}{2}] & Bi[-\beta^{1/3}\frac{a}{2}] \\
Ai[-(-\beta)^{1/3}\frac{a}{2}] & Bi[-(-\beta)^{1/3}\frac{a}{2}] & -Ai[-\beta^{1/3}\frac{a}{2}] & -Bi[-\beta^{1/3}\frac{a}{2}] \\
(-\beta)^{1/3}Ai'[-(-\beta)^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Bi'[-(-\beta)^{1/3}\frac{a}{2}] & \beta^{1/3}Ai'[-\beta^{1/3}\frac{a}{2}] & \beta^{1/3}Bi'[-\beta^{1/3}\frac{a}{2}] \\
(-\beta)^{1/3}Ai'[-(-\beta)^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Bi'[-(-\beta)^{1/3}\frac{a}{2}] & -\beta^{1/3}Ai'[-\beta^{1/3}\frac{a}{2}] & -\beta^{1/3}Bi'[-\beta^{1/3}\frac{a}{2}]
\end{bmatrix}
\begin{bmatrix} E_5 \\ E_6 \\ E_7 \\ E_8 \end{bmatrix}
$$

$$
= \begin{bmatrix}
Ai[\beta^{1/3}\frac{a}{2}] & Bi[\beta^{1/3}\frac{a}{2}] & Ai[(-\beta)^{1/3}\frac{a}{2}] & Bi[(-\beta)^{1/3}\frac{a}{2}] \\
Ai[\beta^{1/3}\frac{a}{2}] & Bi[\beta^{1/3}\frac{a}{2}] & -Ai[(-\beta)^{1/3}\frac{a}{2}] & -Bi[(-\beta)^{1/3}\frac{a}{2}] \\
\beta^{1/3}Ai'[\beta^{1/3}\frac{a}{2}] & \beta^{1/3}Bi'[\beta^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Ai'[(-\beta)^{1/3}\frac{a}{2}] & (-\beta)^{1/3}Bi'[(-\beta)^{1/3}\frac{a}{2}] \\
\beta^{1/3}Ai'[\beta^{1/3}\frac{a}{2}] & \beta^{1/3}Bi'[\beta^{1/3}\frac{a}{2}]] & -(-\beta)^{1/3}Ai'[(-\beta)^{1/3}\frac{a}{2}] & -(-\beta)^{1/3}Bi'[(-\beta)^{1/3}\frac{a}{2}]
\end{bmatrix}
\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}
$$

In the Region 4,

$$
\begin{aligned}
\frac{d^2x}{du^2} &= 0 \\
x(u) &= v^x_{\text{final}}\left(u - \frac{a}{2}\right) + x_{\text{final}} \\
\dot{x}(u) &= v^x_{\text{final}}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^2y}{du^2} &= 0 \\
y(u) &= v^y_{\text{final}}\left(u - \frac{a}{2}\right) + y_{\text{final}} \\
\dot{y}(u) &= v^y_{\text{final}}
\end{aligned}
$$

where (x_{final}, y_{final}) and $(v^x_{final}, v^y_{final})$ are final position and velocity of the test mass respectively at $u = \frac{a}{2}$.

Because of continuity relation of position and velocity vector between Region 3 and 4, at $u' = \frac{a}{2}$ we get

$$
\begin{aligned}
x_{\mathrm{final}} &= \frac{1}{2}\left(E_5 Ai(0) + E_6 Bi(0) + E_7 Ai(0) + E_8 Bi(0)\right) \\
y_{\mathrm{final}} &= \frac{1}{2}\left(E_5 Ai(0) + E_6 Bi(0) - E_7 Ai(0) - E_8 Bi(0)\right) \\
v^x_{\mathrm{final}} &= \frac{1}{2}\left((-\beta)^{1/3} E_5 Ai'(0) + (-\beta)^{1/3} E_6 Bi'(0) + \beta^{1/3} E_7 Ai'(0) + \beta^{1/3} E_8 Bi'(0)\right) \\
v^y_{\mathrm{final}} &= \frac{1}{2}\left((-\beta)^{1/3} E_5 Ai'(0) + (-\beta)^{1/3} E_6 Bi'(0) - \beta^{1/3} E_7 Ai'(0) - \beta^{1/3} E_8 Bi'(0)\right)
\end{aligned}
$$

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